Theorem (≈6.2.3 Puterman): Suppose L is a contraction mapping, then for arbitrary v^0 the sequence $\{v^0, \ldots, v^n\}$ defined by $v^{n+1} = Lv^n$ converges to v^* as $n \to \infty$, where $v^* = Lv^*$.

Proof:

First, show that $\{v^0, \ldots, v^n\}$ converges to a limit point. For some n:

$$|v^{n+1} - v^n| = |L^n v^1 - L^n v^0| \leq \lambda^n |v^1 - v^0|$$

Therefore it follows that $Lim_{n\to\infty}|v^{n+1} - v^n| = 0$ and the sequence converges to a limit point that we will call v^*.

Next, show that limit point v^* is the solution to the optimality equations. This follows because

$$\begin{aligned} 0 \leq |Lv^* - v^*| &\leq |Lv^* - v^n + v^n - v^*| \\ &\leq |Lv^* - v^n| + |v^n - v^*| \\ &\leq \lambda|v^* - v^{n-1}| + |v^n - v^*| \end{aligned}$$

The first inequality follows from the *triangle inequality* and the second follows from L being a *contraction mapping* (this is proved separately in Theorem 6.2.4). From the first part of the proof we have that $Lim_{n\to\infty}|v^n - v^*| = 0$ it follows that in the limit that $n \to \infty$ the right hand side of the above inequality goes to zero. Therefore

$$0 \leq |Lv^* - v^*| \leq 0$$

and it follows that the limiting point v^* is the solution to the optimality equations $Lv = v$.