TSP Example, Lecture 4

Following is the dynamic programming formulation of the travelling salesperson problem (TSP) discussed in lecture 4 (see Dreyfus and Law for a more detailed description).

Let $N_j = \{2,3, \dots, j-1, j+1, \dots, N\}$ be the set of all locations except location j and the starting location, 1, and let S_i index any subset of N_j with i members

States: defined by current location, j , and subset of cities visited, S_i

Action: location to arrive from among the set of locations already visited

Optimality equations:

$$v_i(j, S_i) = \min_{a_i \in S_i}\{d_{a_i j} + v_{i-1}(a_i, S_i - a_i)\}, \quad \forall j, \forall S_i \subset N_j$$

$$v_0(j, \emptyset) = d_{1j}, \forall j$$

Consider the instance of TSP define by the following distance matrix for 5 locations:

i/j	1	2	3	4	5
1	0	3	1	5	4
2	1	0	5	4	3
3	5	4	0	2	1
4	3	1	3	0	3
5	5	2	4	1	0

Following is the solution:

Stage 0: The following value function estimates are based on the distance from the starting location, 1, to the current location, given that no previous locations were visited in between. This defines the boundary condition for the problem.

$$v_0(2, \emptyset) = d_{12} = 3$$

$$v_0(3,\emptyset\,) = d_{13} = 1$$

$$v_0(4,\emptyset\,) = d_{14} = 5$$

$$v_0(5,\emptyset\,) = d_{15} = 4$$

Stage 1: The following value function estimates are generated for each possible current location, i.e., 2,3,4,5, for each combination of the current location with the prior location visited that one location was visited prior to the current one.

$$v_1(2,\{3\}) = d_{32} + v_0(3,\emptyset) = 4 + 1 = 5$$

$$v_1(2,\{4\}) = d_{42} + v_0(4,\emptyset) = 1 + 5 = 6$$

$$v_1(2,\{5\}) = d_{52} + v_0(5,\emptyset) = 2 + 4 = 6$$

$$v_1(3,\{2\}) = d_{23} + v_0(2,\emptyset) = 5 + 3 = 8$$

$$v_1(3,\{4\}) = d_{43} + v_0(4,\emptyset) = 3 + 5 = 8$$

$$v_1(3,\{5\}) = d_{53} + v_0(5,\emptyset) = 4 + 4 = 8$$

$$v_1(4,\{2\}) = d_{24} + v_0(2,\emptyset) = 4 + 3 = 7$$

$$v_1(4,\{3\}) = d_{34} + v_0(3,\emptyset) = 2 + 1 = 3$$

$$v_1(4,\{5\}) = d_{54} + v_0(5,\emptyset) = 1 = 4 = 5$$

$$v_1(5,\{2\}) = d_{25} + v_0(2,\emptyset) = 3 + 3 = 6$$

$$v_1(5,\{3\}) = d_{35} + v_0(3,\emptyset) = 1 + 1 = 2$$

$$v_1(5,\{4\}) = d_{45} + v_0(4,\emptyset) = 3 + 5 = 8$$

Stage 2: This is the first stage were we have more than one prior location that we may have arrive from, so we have to compare to get the minimum distance.

$$v_2(2,\{3,4\}) = \min\{d_{32} + v_1(3,\{4\}),\ \ d_{42}v_1(4,\{3\}\}) = \min\{12,4\} = 4$$

$$v_2(2,\{3,5\}) = \min\{d_{32} + v_1(3,\{5\}),\ d_{52}v_1 5,\{3\}\} = \min\{12,4\} = 4$$

$$v_2(2,\{4,5\}) = \min\{d_{42} + v_1(4,\{5\}),\ d_{52}v_1(5,\{4\}\} = \min\{6,10\} = 6$$

And the remaining optimal value function results are;

$$v_2(3,\{2,4\}) = 10$$

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$$v_2(3,\{4,5\}) = 8$$

$$v_2(4,\{2,3\}) = 9$$

$$v_2(4,,\{2,5\}) = 7$$

$$v_2(4,\{3,5\}) = 3$$

$$v_2(5,\{2,3\}) = 8$$

$$v_2(5,\{2,4\}) = 9$$

$$v_2(5,\{3,4\}) = 6$$

State 3: In this case there are 3 possible prior locations that we could arrive from.

$$v_3(2,\{3,4,5\}) = \min\{d_{32} + v_2(3,\{4,5\}), d_{42} + v_2(4,\{3,5\}), d_{52} + v_2(5,\{3,4\}\}$$

$$v_3(3,\{2,4,5\}) = 10$$

$$v_3(4,\{2,3,5\}) = 8$$

$$v_3(5,\{2,3,4\}) = 7$$

Stage 4:

$$v_4(\emptyset,\{2,3,4,5\} =$$

$\min\{d_{21} + v_3(2,\{3,4,5\}), d_{31} + v_3(3,\{2,4,5\}), d_{41} + v_3(4,\{2,3,5\}), d_{51} + v_3(5,\{2,3,4\}\}$ =5

The optimal tour has total length 5 and it is constructed by working backward through the problem to obtain 1→ 3→ 5→ 4→ 2→ 1