Lecture 3 Exampled

This short document provides detailed solutions to the ERP problem using the two different (but equivalent) model formulations discussed in lecture 3:

Standard Formulation:

States: age of machine at stage t, s_t

Actions: $a_t \in \{Buy,\ Keep\}$

Rewards:

- $c(s_t) =$ cost of operating a machine of age s_t
- $p =$ price of new machine
- $r(s_t) =$ trade-in value for machine of age s_t

In this formulation it is assumed that purchase decisions are made at the start of each stage. Operating costs and trade-in reward are obtained at the end of the stage (before the start of the next stage).

Optimality Equations:

$$v_t(s_t) = \min_{a_t \in \{B,K\}} \{p - r(s_t) + c(0) + v_{t+1}(1),\ c(s_t) + v_{t+1}(s_t + 1)\}, \quad \forall s_t$$

Boundary Condition:

$$v_T(s_T) = -\mathrm{r}(\mathrm{s_t}), \quad \forall s_T$$

The optimality equations can be solved using backwards recursion (also called *backwards induction*). We can start with the last period, using the boundary condition.

<u>Stage 6:</u> $v_6(1) = -7, v_6(2) = -6, v_6(3) = -2, v_6(4) = -1, v_6(5) = 0$

<u>Stage t = 5 (start of year 5):</u>

$v_5(1) = \min\{14 - 7 - 7, 4 - 6\} = -2, \quad a_5\,(1) = K$

$v_5(2) = \min\{14 - 6 - 7, 5 - 2\} = 1, \quad a_5\,(2) = B$

$v_5(3) = \min\{14 - 2 - 7, 9 - 1\} = 5, \quad a_5\,(3) = B$

$v_5(4) = \min\{14 - 1 - 7, 12 - 0\} = 6, \quad a_5\,(4) = B$

$v_5(5) = 14 - 7 = 7, \quad a_5\,(5) = B$

Stage t=4:

$v_4(1) = \min\{\,14 - 7 - 2, 4 + 1\,\} = 5$, $a_4\,(1) =\ B\ or\ K$

$v_4(2) = \min\{\,14 - 6 - 2, 5 + 5\,\} =\ 6,$ $a_4\,(2) =\ B$

$v_4(3) = \min\{14 - 2 - 2, 9 + 6\ \} =\ 10\,,$ $a_4\,(3) =\ B$

$v_4(4) = \min\{\,14 - 1 - 2, 12 + 7\,\} =\ 11\,,$ $a_4\,(4) =\ B$

Stage t = 3:

$v_3(1) = \min\{14 - 7 + 5, 4 + 6\ \} =\ 10\,,$ $a_3\,(1) =\ K$

$v_3(2) = \min\{14 - 6 + 5, 5 + 10\ \} =\ 13\ ,$ $a_3\,(2) =\ B$

$v_3(3) = \min\{14 - 2 + 5, 9 + 11\ \} =\ 17\,,$ $a_3\,(3) =\ B$

Stage 2:

$v_2(1) = \min\{\,14 - 7 + 10, 4 + 6\,\} =\ 17\,,$ $a_2\,(1) =\ B\ or\ K$

$v_2(2) = \min\{\,14 - 6 + 10, 5 + 17\,\} =\ 18\,,$ $a_2\,(2) =\ B$

Stage 1: , $a_1\,(1) =\ B$ (by assumption)

Total Cost = 12+2+ $v_2(1)$ = 31

Optimal Policy

Because there were ties there are multiple optimal policies. Reviewing the optimal actions and the implied state transitions yields the following optimal policies.

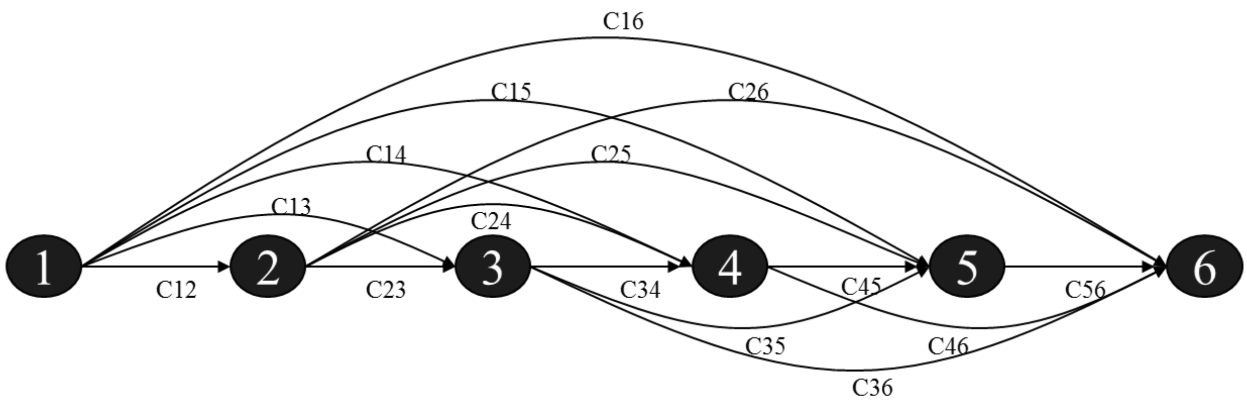
Optimal Policy 1: $a_1(1) = B, a_2(1) = B, a_3(1) = K, a_4(2) = B, a_5(1) = K$

Optimal Policy 2: $a_1(1) = B, a_2\,(1) = K, a_3(2) = B, a_4(1) = K, a_5(2) = B$

Optimal Policy 3: $a_1(1) = B, a_2(1) = K, a_3(2) = B, a_4(1) = B, a_5(1) = K$

Shortest Path Formulation

We also considered a second formulation of the equipment replacement problem as a *shortest path problem* (see slide 11, lecture 3). In this formulation the problem is represented by a directed graph. Following is for the example from lecture 3:



The optimality equations for an n period problem are:

$$v_i = \min_{j=i+1,\ldots,n} \{c_{ij} + v_j\}, \qquad v_n = 0$$

Thanks to the Theorems from Wagner and Whitin discussed in Lecture 3 it is only optimal to produce when there is zero inventory. Therefore each stage has a single state (zero inventory). The optimal action is how many periods of demand to produce for.

c_{ij} is the total cost given new purchase in period i and j

c_{ij}= operating cost during years i,i+1,…,j-1 + purchase cost at the beginning of year i - trade-in value at the beginning of year j

We can precompute all the costs used in the optimality equations.

Costs for arcs leaving node 1:
c_{12}=2+12-7=7
c_{13}=2+4+12-6=12
c_{14}=2+4+5+12-2=21
c_{15}=2+4+5+9+12-1=31
c_{16}=2+4+5+9+12+12-0=44

Costs for arcs leaving node 2:
c_{23}=2+12-7=7
c_{24}=2+4+12-6=12
c_{25}=2+4+5+12-2=21
c_{26}=2+4+5+9+12-1=31

Costs for arcs leaving node 3:
c_{34}=2+12-7=7
c_{35}=2+4+12-6=12
c_{36}=2+4+5+12-2=21

Costs for arcs leaving node 4:
c_{45}=2+12-7=7
c_{46}=2+4+12-6=12

Costs for arcs leaving node 5:
c_{56}=2+12-7=7

Solution to Optimality Equations:

$v_6 = 0$

$v_5 = 7$

$v_4 = \min\{c_{45} + v_5, c_{46}\} = 12$

$v_3 = \min\{c_{34} + v_4, c_{35} + v_5, c_{36}\}$= 19

$v_2 = \min\{c_{23} + v_3, c_{24} + v_4,\ c_{25} + v_5\ c_{26}\}$= 24

$v_1 = \min\{c_{12} + v_2, c_{13} + v_3, c_{14} + v_4, c_{15} + v_5, v_6\}$ = 31