Shortest Path on a Cyclic Network, Lecture 7

This example covers how to solve a shortest path problem on a network with cycles. Since there is no consistent ordering of nodes that is known a priori the algorithm selects nodes sequentially as the problem is solved. The algorithm for this problem is as follows.

Shortest Path Algorithm

Select a start and end node and label all nodes in the network.

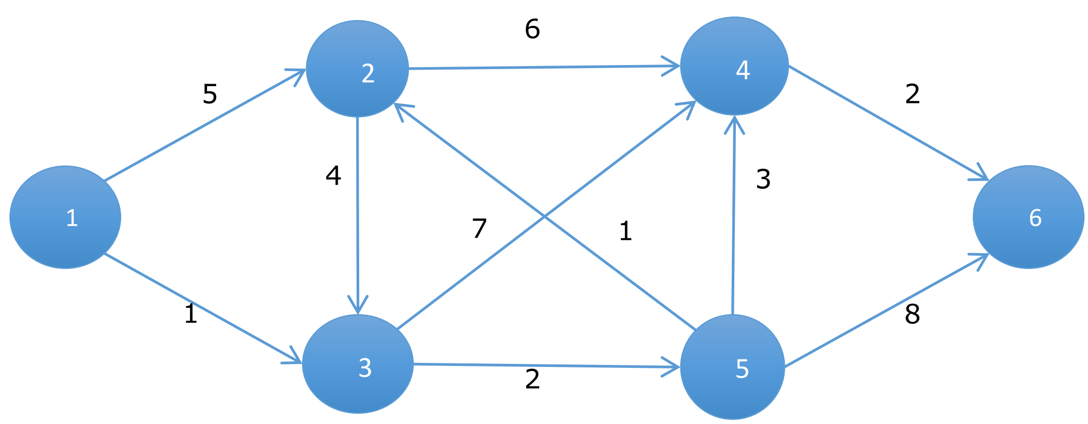
Boundary Condition: For stage 1 set $N_1 = \{1\}$, $v_1(j) = d_{1j}$. Note that $d_{1j} = \infty$ if there is no arc from node 1 to j. Set i=2.

1. Find $k_i = \underset{j \notin N_{i-1}}{\operatorname{argmin}} v_{i-1}(j)$.
2. Set $N_{i-1} \cup \{k_i\}$.
3. Compute $v_i(j)$ as:

$$v_i(j) = \begin{cases} v_{i-1}(j) & if \quad j \in N_i \\ \min\{v_{i-1}(j),\ v_{i-1}(k_i) + d_{k_i,\,j}\} & if\ j \notin N_i \end{cases}$$

If N_i contains all but the last (end) node then stop. Otherwise set i = i+1 and return to step 1. At each step, i, one new node (node k_i, selected in step 1) is added to the set of “visited” nodes, N_i (in step 2). Once a node is added to N_i the shortest path between it and node 1 will not change. In step 3 all nodes not already in N_i are evaluated to see if the addition of the new node, k_i, allows a shorter path.

Consider the following specific example.



Assume the goal is to find the shortest path from node 1 to node 6. The numeric node labels, {1,2,3,4,5,6}, are arbitrarily selected.

Boundary Condition: For stage 1 initiate the set of i closest nodes to node 1 to be $N_1 = \{1\}$, and the optimal value function to $v_1(j) = d_{1j}$. Note that $d_{1j} = \infty$ if there is no arc from node 1 to j. Thus, for stage 1 we have the following.

Stage 1: $N_1 = \{1\},\ k_1 = 1,\ v_1(j) = d_{1j}$

$$v_1(1) = 0, v_1(2) = 5, v_1(3) = 1, v_1(4) = v_1(5) = v_1(6) = \infty$$

Stage 2:

Step 1: $k_2 = argmin_{j \notin N_1}\{v_1(j)\} = 3$

Step 2: $N_2 = \{1{,}3\}$

Step 3:

$$v_2(1) = 0, v_2(3) = 1,$$

$$v_2(2) = \min\{v_1(2), v_1(3) + d_{32}\} = \min\{5, \infty\} = 5,$$

$$v_2(4) = \min\{v_1(4), v_1(3) + d_{34}\} \min\{\infty, 8\} = 8,$$

$$v_2(5) = \min\{v_1(5), v_1(3) + d_{35}\} \min\{\infty, 3\} = 3,$$

$$v_2(6) = \min\{\infty, \infty\} = \infty,$$

Stage 3:

Step 1: $k_3 = argmin_{j \notin N_2}\{v_2(j)\} = 5$

Step 2: $N_2 = \{1{,}3{,}5\}$

Step 3:

$$v_3(1) = 0, v_3(3) = 1, v_3(5) = 3$$

$$v_3(2) = \min\{v_2(2), v_2(5) + d_{52}\} = \min\{5{,}4\} = 4,$$

$$v_3(4) = \min\{v_2(4), v_2(5) + d_{54}\} = \min\{8{,}6\} = 6,$$

$$v_3(6) = \min\{v_2(6), v_2(5) + d_{56}\} = \min\{\infty, 11\} = 11,$$

Stage 4:

Step 1: $k_4 = argmin_{j \notin N_3}\{v_3(j)\} = 2$

Step 2: $N_4 = \{1{,}2{,}3{,}5\}$

Step 3:

$$v_4(1) = 0, v_4(2) = 4, v_4(3) = 1, v_4(5) = 3$$

$$v_4(4) = \min\{v_3(4), v_3(2) + d_{24}\} = \min\{6{,}10\} = 6,$$

$$v_4(6) = \min\{v_3(6), v_2(6) + d_{26}\} = \min\{11, \infty\} = 11,$$

Stage 5:

Step 1: $k_5 = argmin_{j \notin N_4}\{v_4(j)\} = 4$

Step 2: $N_2 = \{1,2,3,4,5\}$

Step 3:

$$v_5(1) = 0, v_5(2) = 4, v_5(3) = 1, v_5(4) = 6, v_5(5) = 3$$

$$v_5(6) = \min\{v_4(6), v_4(4) + d_{46}\} = \min\{11,8\} = 8$$

The shortest path length is 8. Backtracking through the solution to find the optimal action at each stage gives the optimal path: 1→ 3→ 5→ 4→ 6.