Theorem (≈6.2.4 Puterman): If $0 < \lambda < 1$ then L is a contraction mapping

Proof:

Let u and v be arbitrary vectors. Consider an arbitrary state s such that $Lv(s) \geq Lu(s)$ and

$$a^* \in argmax_a\{r(s,a) + \lambda \sum_j p(j|s,a)v(j)\}$$

Then it follows that

$$0 \leq Lv(s) - Lu(s) \leq r(s,a^*) + \lambda \sum_j p(j|s,a^*)v(j) - r(s,a^*) - \lambda \sum_j p(j|s,a^*)u(j)$$

$$\leq \lambda \sum_j p(j|s,a^*)(v(j) - u(j))$$

$$\leq \lambda \sum_j p(j|s,a^*)|v - u| = \lambda|v - u|$$

Thus $Lv(s) - Lu(s) \leq \lambda|v - u|$for all s. Repeating this for the case in which $Lv(s) \leq Lu(s)$ yields $|Lv(s) - Lu(s)| \leq \lambda|v - u|$ for all s. Therefore the results follows that $|Lv - Lu| \leq \lambda|v - u|$ since the property is true for all states, s. Therefore L is a contraction mapping.