Knapsack Example, Lecture 4

Following is the solution to the "knapsack problem" example discussed in Lecture 4. The knapsack problem is a special case of the "Resource Allocation Problem" discussed in class. See the lecture notes for the model formulation and definition of notation used here.

Let $v_t(s_t)$ be the maximum benefit that can be earned for item t given an s_t-pound knapsack assuming the optimal policy is followed for stage t and all future stages. Suppose the knapsack has a capacity of 10lb and is to be filled with the following items:

Item 1: 4 lbs, Benefit 11

Item 2: 3 lbs, Benefit 7

Item 3: 5 lbs, Benefit 12

Therefore:

$$r_1(a_1) = 11a_1, \quad r_2(a_2) = 7a_2, \quad r_3(a_3) = 12a_3\ ,$$

$$f_1(a_1) = 4a_1, \quad f_2(a_2) = 3a_2, \quad f_3(a_3) = 5a_3$$

To find the optimal policy, i.e., number of each item to put in the knapsack, we must solve the optimality equations. Note: You can order the items any way you want, and you will get the same optimal policy. Here I have opted to order the items 1, 2, 3. Starting with the last "stage," i.e., the last item listed above, item 3, we have the following optimality equations:

$v_3(s_3) = \max\{12a_3\}$ where $5a_3 \le s_3$ and a_3 is an integer number of items to put in the knapsack. It follows that in stage 3: $v_3(10) = a_3 \times 12 = 2 \times 12 = 24,\ v_3(5) = v_3(6) = v_3(7) = v_3(8) = v_3(9) = a_3 \times 12 = 1 \times 12 = 12,\ v_3(0) = v_3(1) = v_3(2) = v_3(3) = v_3(4) = a_3 \times 12 = 0 \times 12$

Optimal decisions in stage 2, i.e., for item 2, are governed by:

$v_2(s_2) = \max\{7a_2 + v_3(s_2 - 3a_2)\}$ where a_2 is an integer satisfying $3a_2 \le s_2$

Thus, we have the following solutions:

$$v_2(10) = \max_{a_2 \in\{0,1,2,3)} \{7(0) + v_3(1) = 24, 7(1) + v_3(7) = 19, 7(2) + v_3(4) = 14, 7(3) + v_3(1) = 21\}$$

$$= 24, \ \ a_2^* = 0$$

$$v_2(9) = \max_{a_2 \in\{0,1,2,3)} \{7(0) + v_3(9) = 12, 7(1) + v_3(6) = 19, 7(2) + v_3(3) = 14, 7(3) + v_3(0) = 21\}$$

$$= 21, \ \ a_2^* = 3$$

Continuing the remaining optimal value functions similarly for $s_2 = 8,7,6, \ldots ,0$ are as follows:

$v_2\ (8) = 19, a_2^* = 1$

$v_2\ (7) = 14, a_2^* = 2$

$v_2\ (6) = 14, a_2^* = 2$

$v_2\ (5) = 12, a_2^* = 1$

$v_2\ (4) = 7, a_2^* = 1$

$v_2\ (3) = 7, a_2^* = 1$

$v_2\ (2) = 0, a_2^* = 0$

$v_2\ (1) = 0, a_2^* = 0$

$v_2\ (0) = 0, a_2^* = 0$

Optimal decisions in stage 1 are governed by:

$v_1(s_1) = \max\{11a_1 + v_2(s_1 - 4a_1)\}$ where a_3 is an integer satisfying $4a_1 \leq s_1$

In stage 1 we begin with all 10 units of capacity available. Therefore we only evaluate the state $s_1 = 10$ as follows:

$$v_1\ (10) = \max_{a_1 \in \{0,1,2\}} \{11 \times 0 + v_2(10, 11(1) + v_2(6), 11(2) + v_2(2) = 22\} = 25, \quad a_1^* = 1$$

Inspecting the results above and backtracking through the optimality equation yields the following optimal policy: $a_1^* = 1$, $a_2^* = 2, a_3^* = 0$, and the total reward is $v_1(1) = 25$.

Exercise: To test your understanding of the approach try solving this problem with a different ordering of items (e.g. 3, 1, 2) and confirm you get the same solution.