Stochastic Programming
Lecture 8

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September 28, 2010

Today's Class

1. Research Paper Presentations
2. Decomposition
3. Optimality Cuts
4. Infeasibility Cuts
5. Algorithm
6. L-Shaped Method

Outline

1. Research Paper Presentations
2. Decomposition
3. Optimality Cuts
4. Infeasibility Cuts
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6. L-Shaped Method

Research Paper Presentations

Starting in mid October we will begin reviewing research papers:

- Statistical sampling
- Multi-Stage Stochastic Linear Programming
- Bounding methods
- Integer L-shaped Method (Piper, ?)
- Probabilistic Constraints (Dr. Uzsoy)
- Robust Optimization
- Stochastic Decomposition

You can work in groups of 1 or 2.

Let me know your topic and who you want to work with by Friday.

Outline

1. Research Paper Presentations
2. Decomposition
   - 2-SLP
   - Bender's Decomposition
3. Optimality Cuts
4. Infeasibility Cuts
5. Algorithm
6. L-Shaped Method

2-SLP

Two-Stage Recourse Problem (2-SLP):

$$\min\{cx + Q(x) \mid Ax = b, x \geq 0\}$$

- Computational Problem: as the number of scenarios grows computing the *recourse function*, $Q(x)$, is hard
- There are many algorithms that take advantage of the special structure of 2-SLPs:
    - L-Shaped Method
    - Dantzig Wolf Decomposition
    - Specialized Methods for Basis Factorization
    - Bunching...

Bender's Decomposition

Bender's Decomposition: Consider the following (deterministic) LP....

$$
\begin{aligned}
&\min\ z = cx + fy \\
&\text{s.t.} \\
&\quad Ax + By \geq b \\
&\quad x \geq 0,\ y \geq 0
\end{aligned}
$$

for a fixed $x = \bar{x}$ we would solve...

$$
\begin{aligned}
&\min fy \\
&\text{s.t.} \\
&\quad By \geq b - A\bar{x} \\
&\quad y \geq 0
\end{aligned}
$$

This problem may be easy (e.g. a network flow problem)

Bender's Decomposition

If the problem is easy for fixed x then it may be (computationally) advantageous to solve the following....

$$\min\{cx + Q(x)|x \geq 0\}$$

where

$$Q(x) = \min\{fy|By \geq b - Ax, y \geq 0\}$$

$Q(x)$ is convex, continuous, but not necessarily differentiable

In the context of SP Bender's decomposition is called the L-shaped method:

Van Slyke, R.M, Wets, R. 1969, L-Shaped Linear Programs with Applications to Optimal Control and Stochastic Programming, SIAM Journal on Applied Mathematics, 17(4), 638-663.

Decomposition

Replace $Q(x)$ with a variable θ and solve the relaxation (master problem):

$$\min\{cx + \theta \,|x \geq 0, \text{optimality cuts}, \ \text{infeasibility cuts}\}$$

Decomposition

Replace $Q(x)$ with a variable θ and solve the relaxation (master problem):

$$\min\{cx + \theta \,|x \geq 0, \text{optimality cuts}, \text{ infeasibility cuts}\}$$

Optimality cuts are supporting hyperplanes of the *epigraph* of $Q(x)$. Consider the dual of $Q(x)$....

$$Q(x) = \max\{\pi(b - Ax) \mid \pi B \leq f, \pi \geq 0\}$$

for a particular $x = \bar{x}$ the following is a supporting hyperplane of $Q(x)$ at $\bar{x}$:

$$\theta \geq \pi(b - Ax)$$

Why?

Outline

Optimality Cuts

Optimality cuts, $\theta \geq \pi(\bar{x})(b - By)$, follow from the subgradient inequality where π is the solution to the dual of the following primal problem:

$$\min\{fy \mid By \geq b - Ax, y \geq 0\}$$

taking the dual it follows that for a particular $x = \bar{x}$:

$$Q(\bar{x}) = \max\{\pi(b - A\bar{x}) \mid \pi B \leq f, \pi \geq 0\}$$

From the subgradient inequality we have:

$$Q(x) \geq Q(\bar{x}) + \pi(\bar{x})((b - Ax) - (b - A\bar{x})) = \pi(\bar{x})(b - Ax)$$

Therefore $\theta \geq \pi(\bar{x})(b - Ax)$ is a valid lower bound on $Q(x)$ for all x

.

Outline

Infeasibility Cuts

Feasibility is represented by K_2. When the master problem produces an $\bar{x}$ that is infeasible an *infeasibility cut* is added to the master problem.

Feasibility is tested by solving the following (Phase I) LP:

$$\begin{aligned}
&\min \; w = ev^{+} + ev^{-} \\
&\text{s.t.} \\
&\quad By + Iv^{+} - Iv^{-} = b - Ax \\
&\quad y, v^{+}, v^{-} \geq 0
\end{aligned}$$

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&\min \; w = ev^+ + ev^- \\
&\text{s.t.} \\
&\quad By + Iv^+ - Iv^- = b - Ax \\
&\quad y, v^+, v^- \geq 0
\end{aligned}$$

If $w^* > 0$ then a feasibility cut is needed to separate $b - A\bar{x}$ from pos(B).

Infeasibility Cuts

If σ is the dual solution to the Phase I LP then the feasibility cut is...

$$\sigma(b - Ax) \leq 0 \tag{1}$$

This has the properties:

- $\sigma(b - A\bar{x}) > 0$, i.e., $(b - A\bar{x})$ does not satisfy the infeasibility cut
- $\sigma t \leq 0$ for all $t \in pos(B)$, i.e, it does not cut off any of the feasible region

Outline

Algorithm

Bender's Decomposition:

Step 0: $\nu = 1$

Step 1: Construct and solve master problem for

$$x^{\nu} = argmin\{cx + \theta \,|optimality\ cuts, infeasibility\ cuts, \forall \nu, x \in X\}$$

Step 2: Solve Phase I LP for subproblem with $x = x^{\nu}$.
If feasible go to Step 3. If infeasible generate *infeasibility cut*, set $\nu = \nu + 1$, and add to master. Return to Step 1.

Step 3: Solve $w^{\nu} = \min\{fy \mid By \geq b - Ax^{\nu}, y \geq 0\}$. If $\theta^{\nu} \geq w^{\nu}$ then stop with x^{ν} as optimal solution. Otherwise generate *optimality cut*, set $\nu = \nu + 1$, and add to master. Return to Step 1.

Outline

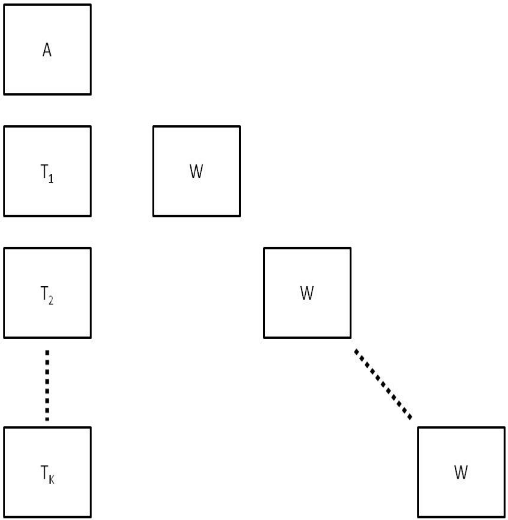
L-Shaped Method

Bender's decomposition is well suited to solving 2-SLPs. Assume finite Ξ with scenarios indexed by $k = 1, ..., K$ with probabilities, p_k. The *extensive form* of 2-SLP is:

$$
\begin{aligned}
&\min cx + \textstyle\sum_{k=1}^{K} p_k q_k y_k \\
&\text{s.t. :} \\
&\quad Ax = b \\
&\quad T_k x + W y_k = h_k, k = 1, ..., K, \\
&\quad x \geq 0, y_k \geq 0, k = 1, ..., K.
\end{aligned}
$$

Where W is the (fixed) recourse matrix. q_k, h_k, T_k, all depend on the scenario k.

L-Shaped Method



L-Shaped Method

The L-shaped method is the most well known method for 2-SLPs. It solves the first stage problem and "outer linearizes" the recourse function, $Q(x)$. To use this method:

- Assume ξ has finite support
- $k = 1, 2, ..., K$ indexes scenarios
- p_k is the probability of scenario k

and *extensive form*:

$$
\begin{aligned}
&\min cx + \textstyle\sum_{k=1}^{K} p_k q_k y_k \\
&\text{s.t.} \\
&\quad Ax = b \\
&\quad T_k x + W y_k = h_k, k = 1, ..., K, \\
&\quad x \geq 0, y_k \geq 0, k = 1, ..., K.
\end{aligned}
$$

L-Shaped Method

Algorithm

- **Step 0**: Set $r = s = \nu = 0$.
- **Step 1**: Set $\nu = \nu + 1$. Solve the master problem:

$$
\begin{aligned}
&\min \;\; cx + \theta \\
&\text{s.t.} \\
&\quad Ax = b, \\
&\quad D_\ell \geq d_\ell, \quad \ell = 1, ..., r, \\
&\quad E_\ell x + \theta \geq e_\ell, \quad \ell = 1, ..., s, x \geq 0, \theta \in \Re.
\end{aligned}
$$

Let (x^ν, θ^ν) be an optimal solution. If no optimality cuts present then $\theta^\nu = -\infty$ and is not considered in computation of x^ν.

L-Shaped Method

- **Step 2**: for $k = 1, ..., K$ solve:

$$\begin{aligned} &\min \ w' = ev^{+} + ev^{-} \\ &\text{s.t.} \\ &\quad Wy + Iv^{+} - Iv^{-} = h_k - T_k x^{\nu}, \\ &\quad y, v^{+}, v^{-} \geq 0 \end{aligned}$$

If for some k, the optimal solution $w' > 0$ stop and generate an *infeasibility cut* as follows. Let σ^{ν} be the associated simplex multiplier and define:

$$D_{r+1} = \sigma_k T_k, \quad d_{r+1} = \sigma_k h_k.$$

Set $r = r + 1$, add the new constraint to the master problem, and return to Step 1. Otherwise if $w' = 0$ for all k then go to Step 3.

L-Shaped Method

- **Step 3**: For $k = 1, ..., K$ solve the linear program

$$\begin{aligned} &\min \; w = q_k y \\ &\text{s.t.} \\ &\quad Wy = h_k - T_k x^\nu \\ &\quad y \geq 0 \end{aligned}$$

Let π_k be the optimal solution to the dual. Define:

$$E_{s+1} = \sum_{k=1}^{K} p_k \pi_k T_k, \quad e_{s+1} = \sum_{k=1}^{K} p_k \pi_k h_k.$$

Let $w^\nu = e_{s+1} - E_{s+1} x^\nu$. If $\theta^\nu \geq w^\nu$ stop with x^ν as an optimal solution. Otherwise set $s = s + 1$, add the new constraint (optimality cut) to the master problem, and return to Step 1.