ISE789B/OR791B - Stochastic Programming

Assignment # 3 - Due October 28, 2010

Note: When answering assignment questions show all of your work. Provide a carefully commented version of all code you write to solve each problem.

Question 1 (3 Points): The epigraph of a function is:

$$epi(f) = \{(x, a) \mid x \in \Re^n, \alpha \in \Re, \alpha \geq f(x)\}$$

Prove that the epigraph of a convex function is a convex set. Explain why this important for the L-shaped method.

Answer: Given two points $(x^1, \alpha^1) \in epi(f)$ and $(x^2, \alpha^2) \in epi(f)$ show that $(\lambda x^1 + (1-\lambda)x^2, \lambda\alpha^1 + (1-\lambda)\alpha^2) \in epi(f), \forall \lambda \in [0, 1]$.

By definition of $epi(f)$ we have $\alpha^1 \geq f(x^1)$ and $\alpha^2 \geq f(x^2)$. Aggregating these two inequalities we have $\lambda\alpha^1 + (1-\lambda)\alpha^2 \geq \lambda f(x^1) + (1-\lambda)f(x^2)$. From convexity of f we have $\lambda f(x^1) + (1-\lambda)f(x^2) \geq f(\lambda x^1 + (1-\lambda)x^2)$. Therefore $\lambda\alpha^1 + (1-\lambda)\alpha^2 \geq f(\lambda x^1 + (1-\lambda)x^2)$ and $(\lambda x^1 + (1-\lambda)x^2, \lambda\alpha^1 + (1-\alpha^1)\alpha^2) \in epi(f), \forall \lambda \in [0, 1]$.

This is important for the L-shaped method since it generates supporting hyperplanes of $epi(Q(x))$. The supporting hyperplanes are valid optimality cuts for all x by the *subgradient inequality*, which requires convexity of $epi(Q(x))$.

Question 2 (3 Points): Use non-anticipativity constraints to prove $WS \leq RP$.

Answer: This proof required recognizing the RP can be formulated as a series of independent subproblems (the WS problem) and an additional set of non-anticipativity constraints (e.g. $x(\omega) = x(\omega'), \forall(\omega, \omega') \in \Omega$). Relaxing the non-anticipativity constraints leaves the WS problem, and therefore $WS \leq RP$.

Question 3 (3 Points): Construct a 2-SLP with stochastic cost coefficients, q, that violates Proposition 2, Section 4.3 B&L. Why does it violate the proposition?

Answer: Following is one example of a 2-SLP that violates the proposition:

$$\min\{2x + E_\xi[\xi y \mid y \geq 1 - x, y \geq 0]\}$$

where ξ takes on values 1 and 3 with probability 3/4, 1/4 respectively. Thus $EV = \min\{2x+1.5(1-x)^+, x \geq 0\} = 1.5 > WS = (3/4)\min\{2x+(1-x)^+, x \geq 0\}+(1/4)\min\{2x+3(1-x)^+, x \geq 0\} = 1.25$. Violation occurs since $Q(x,\xi)$ is concave with respect to $Q(\xi)$ and proof of Proposition 2 requires Jensen's inequality which requires convex $Q(x,\xi)$.

Question 4 (12 Points): Consider the following 2-SLP:

$$\begin{aligned} \min 2x_1 + x_2 + \quad & E_\xi[Q(x,\xi)] \\ & s.t. \\ & x_1 + x_2 \leq 7, \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$Q(x,\xi) = \min\{3y_1+2y_2+3y_3+2y_4 | y_1+2y_2 \geq \xi_1, y_1 \leq x_1, y_2 \leq x_2, y_2 \leq \xi_2, y_3 \geq \xi_2-x_2, y_4 \geq x_2-\xi_2, y_i \geq 0, \forall i\}]$$

where $\xi = \binom{\xi_1}{\xi_2}$ can take values $\binom{3}{2}, \binom{5}{3}, \binom{7}{3}$ with probability 1/3 each.

a. Use Jensen's inequality to compute a lower bound on the optimal solution to the above 2-SLP.

b. Choose the scenario $\binom{7}{3}$ as the reference scenario and compute EVRS.

c. Solve the PAIRS problem for $\binom{3}{2}, \binom{5}{3}, \binom{7}{3}$ as the reference scenario. Compute SPEV.

d. Use the results from (b) and (c) to compute upper and lower bounds on VSS.

Answer: Part (a): Solve the MV problem using $E_\xi[\xi] = 1/3\binom{3}{2} + 1/3\binom{5}{3} + 1/3\binom{7}{3} = \binom{5}{8/3}$. The optimal solution to MV is $z^* = 7.67$ which is a lower bound on the optimal solution to the 2-SLP (from Jensen's inequality).

Part (b): Use $\binom{7}{3}$ as the reference scenario and solve the 2-SLP assuming $\xi = \binom{7}{3}$ with probability 1. Optimal solution to this reference scenario problem is $x_1^r = 1$, $x_2^r = 3$. Solve the 3 subproblems to compute $EVRS = E_\xi[z(x^r, \xi)] = 11.33$.

Part (c): Using $\binom{7}{3}$ as the reference scenario we get $SPEV = 11.33$.

Part (d): From Theorem 9 of Section 4.6 B&L we have $0 \leq EVRS - EPEV \leq VSS \leq EVRS - SPEV$. Since $EVRS - SPEV = 0$ it follows that $0 \leq VSS \leq 0$ and the VSS for this problem is 0.

Question 5 (10 points) Develop code to implement the L-shaped method to solve the above 2-SLP from Question 4. Show all steps including master solutions, and optimality cuts at each iteration. Compute the VSS and compare to the bounds in part (d). Provide carefully commented code.

Answer: The optimal solution to the above SLP is $x_1^* = 1, x_2^* = 3, Q(x^*) = 11.33$. The L-shaped algorithm generates a series of feasibility cuts, followed by optimality cuts that converge to the optimal solution. Cuts vary depending on the order of scenarios in step 2, the starting solution, and due to the fact that the Phase I LP used to generate feasibility cuts has alternative optimal solutions, i.e., multiple choices for feasibility cuts.

Question 6 (10 Points): Solve Q. 2, p. 169, Birge and Louveaux.

Answer: This problem required (a) solving a modified version of the example on p. 159 using the standard L-shaped method (b) solving the same problem using the multi-cut L-shaped method and (c) implementing a hybrid version of the multi-cut method in which scenarios are clustered into 3 groups and 3 optimality cuts are added at each iteration.

For part (b) the multi-cut version requires only 2 iterations, with 7 cuts added to the master LP at each iteration.

For part (c) only 2 iterations are required. Thus the hybrid version of the multi-cut method results in a smaller total number of total iterations (compared to standard L-shaped) and a smaller master LP at each iteration (compared to multi-cut method).

Question 7 (10 Points): Consider the following 2-SLP:

$$\min\{x + E_\xi[Q(x, \xi)] \mid x \geq 0\}$$

where $Q(x, \xi) = \min\{y \mid y + x \geq \xi, 0 \leq y \leq 10\}$ and random variable $\xi \in \{0, 1, 2, 3, 4, ..., 20\}$. Use the L-shaped method to solve this problem by hand. How many *feasibility cuts* will be generated before moving on to step 3 of the L-shaped method?

Answer: The L-shaped method is not very efficient for this problem. If scenarios or ordered as stated above then step 3 will generate feasibility cuts for scenarios $\xi = 0, 1, ..., 9$ of the form $x \geq 1, x \geq 2, ..., x \geq 10$. Computed cuts are of the form $x \geq \xi$ and the master problem must be solved after the addition of each infeasibility cut. Alternatively, as discussed in class, a single induced feasibility cut, $x \geq 10$, could be added a priori to avoid the repeated addition of feasibility constraints. This is equivalent to picking the scenario $\xi = 10$ resulting in the strongest feasibility cut first.