Stochastic Programming
Lecture 9

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Today's Class

1. L-Shaped Method
2. L-Shaped Special Cases
3. L-Shaped Example: Newsvendor Problem
4. "Multi-Cut" L-Shaped Method

Outline

1. L-Shaped Method
2. L-Shaped Special Cases
3. L-Shaped Example: Newsvendor Problem
4. “Multi-Cut” L-Shaped Method

L-Shaped Method

The L-shaped method is the most well known method for 2-SLPs. It is a special case of *Benders Decomposition* that "outer linearizes" the recourse function, $Q(x)$. To use this method:

- Assume ξ has finite support
- $k = 1, 2, ..., K$ indexes scenarios
- p_k is the probability of scenario k

The *extensive form* of 2-SLP is:

$$\begin{aligned}
&\min cx + \textstyle\sum_{k=1}^{K} p_k q_k y_k \\
&\text{s.t.} \\
&\quad Ax = b \\
&\quad T_k x + W y_k = h_k, k = 1, ..., K, \\
&\quad x \geq 0, y_k \geq 0, k = 1, ..., K.
\end{aligned}$$

L-Shaped Method

Algorithm

- **Step 0**: Set $r = s = \nu = 0$.
- **Step 1**: Set $\nu = \nu + 1$. Solve the master problem:

$$\begin{aligned} &\min \; cx + \theta \\ &\text{s.t.} \\ &\quad Ax = b, \\ &\quad D_\ell \geq d_\ell, \;\; \ell = 1, ..., r, \\ &\quad E_\ell x + \theta \geq e_\ell, \;\; \ell = 1, ..., s, x \geq 0, \theta \in \Re. \end{aligned}$$

Let (x^ν, θ^ν) be an optimal solution. If cuts present then $\theta^\nu = -\infty$ and is not considered in computation of x^ν.

L-Shaped Method

- **Step 2**: for $k = 1, ..., K$ solve:

$$\begin{aligned} &\min\ w' = ev^+ + ev^- \\ &\text{s.t.} \\ &\quad Wy + Iv^+ - Iv^- = h_k - T_k x^\nu, \\ &\quad y, v^+, v^- \geq 0 \end{aligned}$$

If for some k, the optimal solution $w' > 0$, stop and generate an *infeasibility cut* as follows. Let σ^ν be the associated simplex multiplier and define:

$$D_{r+1} = \sigma_k T_k, \quad d_{r+1} = \sigma_k h_k.$$

Set $r = r + 1$, add the new constraint to the master problem, and return to Step 1. Otherwise if $w' = 0$ for all k then go to Step 3.

L-Shaped Method

- **Step 3**: For $k = 1, ..., K$ solve the linear program

$$\begin{aligned}
&\min \;\; w = q_k y \\
&\text{s.t.} \\
&\quad Wy = h_k - T_k x^\nu \\
&\quad y \geq 0
\end{aligned}$$

  Let π_k be the optimal solution to the dual. Define:

$$E_{s+1} = \sum_{k=1}^{K} p_k \pi_k T_k, \quad e_{s+1} = \sum_{k=1}^{K} p_k \pi_k h_k.$$

  Let $w^\nu = e_{s+1} - E_{s+1} x^\nu$. If $\theta^\nu \geq w^\nu$ stop with x^ν as an optimal solution. Otherwise set $s = s + 1$, add the new constraint (optimality cut) to the master problem, and return to Step 1.

Outline

L-Shaped Special Cases

Sometimes the L-shaped method can be simplified. For instance, Step 2 is not required:

- The recourse problem has *complete recourse* or *relatively complete recourse*
- A set of *induced constraints* can be added to K_1 which guarantee feasibility in the second stage

Outline

L-Shaped Example: Newsvendor Problem

Example 1, p. 159, B & L

Let $Q(x,\xi) = (\xi - x)^+ + (x - \xi)^+$ and let ξ take on values 1,2,4 with probability 1/3 each. Assume $c = 0$ and $0 \le x \le 10$.

L-Shaped Algorithm:
Step 0: Set $r = s = \nu = 0$.

Step 1: Set $\nu = \nu + 1$. Solve the master problem:

$$\begin{aligned}
&\min \ cx + \theta \\
&\text{s.t.} \\
&\quad Ax = b, \\
&\quad D_\ell \ge d_\ell, \quad \ell = 1, ..., r, \\
&\quad E_\ell x + \theta \ge e_\ell, \quad \ell = 1, ..., s, x \ge 0, \theta \in \Re.
\end{aligned}$$

Step 3: Solve $k = 1, ..., K$ subproblems: $\min\{q_k y \mid h_k - T_k x^\nu, y \ge 0\}$. Let π_k be the optimal solution to the dual. Create optimality cut:

$$E_{s+1} = \sum_{k=1}^{K} p_k \pi_k T_k, \quad e_{s+1} = \sum_{k=1}^{K} p_k \pi_k h_k.$$

Let $w^\nu = e_{s+1} - E_{s+1} x^\nu$. If $\theta^\nu \ge w^\nu$ stop with x^ν as an optimal solution. Otherwise set $s = s + 1$, add the new constraint (optimality cut) to the master problem, and return to Step 1.

L-Shaped Example: Newsvendor Problem

L-Shaped Method, Iteration 1:
Step 1: Master problem is $\min\{\theta \mid 0 \leq x \leq 10\}$. Therefore $\theta^* = -\infty$ and we can arbitrarily select $x^1 = 0$ as a starting point (we could have selected a different starting point (e.g. $x^1 = E[\xi]$) but we'll follow B & L's example).

Step 2: Skip since problem has relatively complete recourse.

L-Shaped Example: Newsvendor Problem

Step 3: The second stage of the newsvendor problem can be written as: $\min\{y^+ + y^- \mid y^+ - y^- = x - \xi, y^+, y^- \geq 0\}$. We need to solve the dual of this to generate the optimality cut:

$$\max\{\pi(x - \xi) \mid \pi \leq 1, \pi \geq -1\}$$

- For $x^1 = 0, \xi = 1$: $y_1^{+*} = 0$, $y_1^{-*} = 1$, $\pi_1 = -1$
- For $x^1 = 0, \xi = 2$: $y_2^{+*} = 0$, $y_2^{-*} = 2$, $\pi_2 = -1$
- For $x^1 = 0, \xi = 4$: $y_3^{+*} = 0$, $y_3^{-*} = 4$, $\pi_3 = -1$

Note: for starting point $x^1 = 0$ all subproblems have the same optimal solution in this iteration.

Optimality cut is $\sum_{k=1}^{3} p_k \pi_k (h_k - T_k x) \leq \theta$.

L-Shaped Example: Newsvendor Problem

Since $h_k = -\xi_k$:

$$E[\pi_k h_k] = 1/3(-1)(-1) + 1/3(-1)(-2) + 1/3(-1)(-4) = 7/3$$

and since T_k = -1 (just a 1×1 matrix in this simple example):

$$E[\pi_k T_k] = 1/3(-1)(-1) + (1/3)(-1)(-1) + (1/3)(-1)(-1) = 1$$

Therefore the optimality cut is: $\theta \geq 7/3 - x$.

Return to step 1 and solve the new master problem:

$$\min\{\theta \mid \theta + x \geq 7/3, 0 \leq x \leq 10\}$$

$\theta^* = -23/3$ and $x^* = 10$. Continue to step 2 and so on until optimality criteria is satisfied (see B&L for remaining steps.

Outline

"Multi-Cut" L-Shaped Method

Algorithm

- **Step 0**: Set $r = s = \nu = 0$.
- **Step 1**: Set $\nu = \nu + 1$. Solve the master problem:

$$\min \; z = cx + \sum_{k=1}^{K} \theta^k$$

s.t.

$$Ax = b$$
$$D_\ell \geq d_\ell, \;\; \ell = 1, ..., r,$$
$$E_{\ell(k)}x + \theta \geq e_{\ell(k)}, \;\; \ell(k) = 1, ..., s(k), k = 1, ..., K, x \geq 0, \theta \in \Re.$$

Let (x^ν, θ^ν) be an optimal solution. If no optimality cuts present then $\theta^\nu = -\infty$ and is not considered in computation of x^ν.

"Multi-Cut" L-Shaped Method

- **Step 2**: for $k = 1, ..., K$ solve:

$$\begin{aligned} &\min \ w' = ev^+ + ev^- \\ &\text{s.t.} \\ &\quad By + Iv^+ - Iv^- = b - Ax \\ &\quad y, v^+, v^- \geq 0 \end{aligned}$$

If for some k, the optimal solution $w > 0$ stop and generate an infeasibility cut as follows. Let σ be the associated simplex multiplier and define:

$$D_{r+1} = \sigma T_k, \quad d_{r+1} = \sigma h_k.$$

Set $r = r + 1$, add the new constraint to the master problem, and return to Step 1. Otherwise if $w' = 0$ for all k then go to Step 3.

"Multi-Cut" L-Shaped Method

- **Step 3**: For $k = 1, ..., K$ solve the linear program

$$\begin{aligned} &\min \;\; w = q_k y \\ &\text{s.t.} \\ &\quad Wy = h_k - T_k x^\nu \\ &\quad y \geq 0 \end{aligned}$$

Let π^k be the optimal solution to the dual of (18). Define:

$$E_{s(k)+1} = p_k \pi_k T_k, \quad e_{s(k)+1} = p_k \pi_k h_k.$$

Optimality Cut: $\theta_k \geq p_k \pi_k (h_k - T_k x)$

Let $w^\nu = e_{s+1} - E_{s+1} x^\nu$. If the optimality cuts are not violated by x^ν for any k then stop with x^ν as an optimal solution. Otherwise set $s = s + 1$, add the new optimality cuts to the master problem, and return to Step 1.

“Multi-Cut” L-Shaped Method

Some important points about the Multi-Cut L-Shaped Method:

- By sending multiple cuts, more detailed information is given to the master. Therefore the number of major iterations is expected to be less than in the single cut method.

- If the number of iterations is reduce sufficiently to compensate for the extra time required to sovle the larger master then the Multi-Cut method can be advantageous.

“Multi-Cut” L-Shaped Method

Some important points about the Multi-Cut L-Shaped Method:

- The Multi-Cut approach is expected to be more effective when the number of realizations K is not significantly larger than the number of first stage constraints m_1.
- Hybrid approaches can be used where subsets of the realizations are grouped to form a smaller number of combination cuts.