Stochastic Programming: Lecture 4

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Today's Class

1. Two-Stage Stochastic Linear Programs with Fixed Recourse
2. News Vendor Problem

Outline

1. Two-Stage Stochastic Linear Programs with Fixed Recourse
2. News Vendor Problem

Two-Stage Stochastic Linear Programs with Fixed Recourse

We'll spend a lot time talking about 2-SLPs with "Fixed Recourse" (Why?). The general formulation is:

$$\begin{aligned}
&\min z = c^T x + E_{\xi(\omega)}[\min q(\omega)^T y(\omega)] \\
&\text{s.t.}: \\
&\quad Ax = b \\
&\quad T(\omega)x + Wy(\omega) = h(\omega), \forall \omega \\
&\quad x \geq 0,\ y(\omega) \geq 0, \forall \omega
\end{aligned}$$

Where W is a known matrix (fixed recourse). $q(\omega) \in \Re^{n_2}$, $h(\omega) \in \Re^{m_2}$, $T(\omega) \in \Re^{m_2 \times n_1}$. $\xi(\omega)$ contains all three stochastic components above and has dimension of $N = n_2 + m_2 + (m_2 \times n_1)$.
Let $\Xi \subseteq \Re^N$ be the support of $\boldsymbol{\xi}$, i.e. the smallest closed subset in $\Re^N$ such that $P\{\xi \in \Xi\} = 1$.

Two-Stage Stochastic Linear Programs with Fixed Recourse

Equivalent formulation (deterministic equivalent problem (DEP)):

$$\begin{aligned}
&\min \;\; z = c^T x + \mathcal{Q}(x) \\
&\text{s.t.}: \\
&\quad Ax = b \\
&\quad x \geq 0
\end{aligned}$$

Where

$$\mathcal{Q}(x) = \mathrm{E}_{\xi}[Q(x, \xi(\omega))]$$

and

$$Q(x, \xi(\omega)) = \min_{y}\{q(\omega)^T y(\omega) | Wy = h(\omega) - T(\omega)x, y(\omega) \geq 0, \forall \omega\}$$

Fixed Recourse and Technology Matrix

When T is deterministic, the formulation can be written as:

$$\begin{aligned} &\min \; z = c^T x + \Psi(\chi) \\ &\text{s.t. :} \\ &\quad Ax = b \\ &\quad Tx - \chi = 0 \\ &\quad x \geq 0 \end{aligned}$$

Where

$$\Psi(\chi) = \mathsf{E}_{\xi}[\psi(\chi, \xi(\omega))]$$

and

$$\psi(\chi, \xi(\omega)) = \min_{y(\omega)} \{q(\omega)^T y(\omega) | Wy(\omega) = h(\omega) - \chi, y \geq 0, \forall \omega\}$$

Outline

News Vendor Problem

Write the news vendor problem as a two-stage stochastic linear program with fixed recourse:

- what are the first and second stage decision variables?
- what is the technology matrix, T?
- what is the recourse matrix, W?

More Examples

- Some health care related examples of two stage stochastic programs

    - Single OR scheduling
    - Multi-OR Surgery Allocation

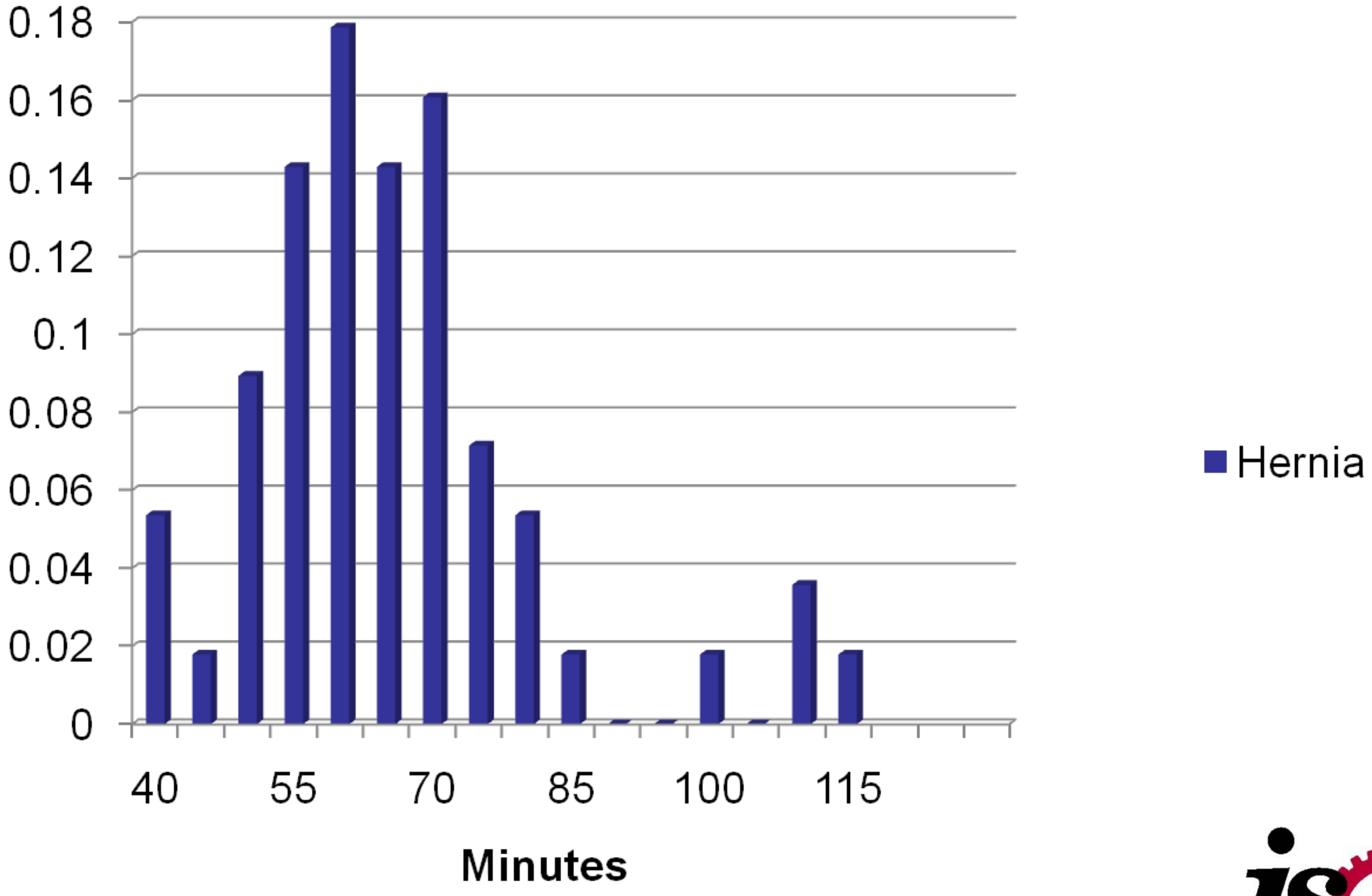
Single OR Scheduling

Problem Description

- *n* patients are scheduled in advance of a day of surgery
- Operating room is available for *d* minutes during the day
- Any time beyond *d* is charged an overtime cost
- The duration of individual surgeries is random

Question: How long should you a lot for each surgery?

Surgery Duration Uncertainty



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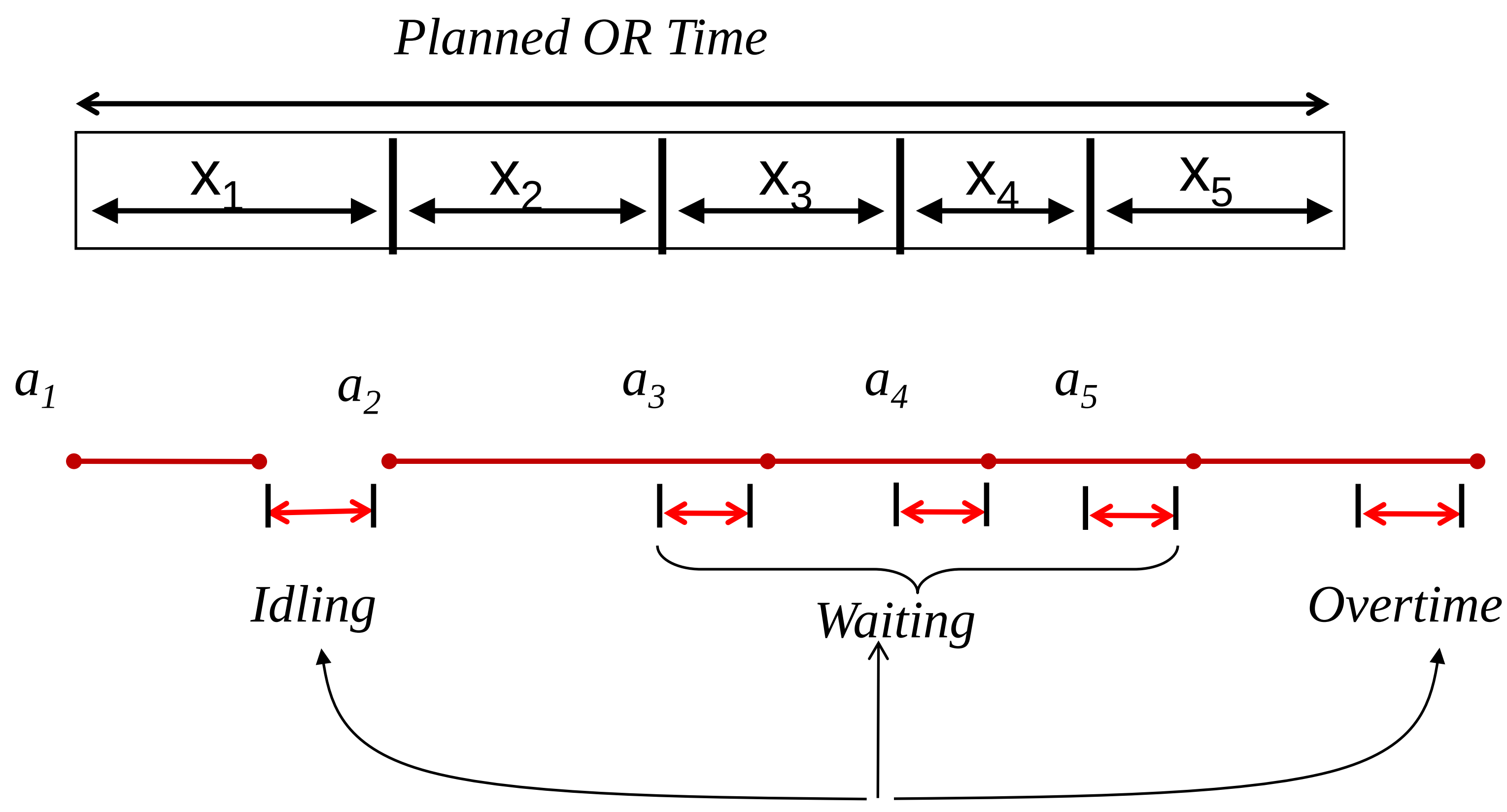
Single OR Scheduling - S(n)/G(n)/1



Min{ Idling + Waiting + Overtime}

Stochastic Optimization Model

Waiting (W), Idling (S), Overtime (L):

$$W_i = \max(W_{i-1} + Z_{i-1}(\omega) - x_{i-1}, 0)$$

$$S_i = \max(-W_{i-1} - Z_{i-1}(\omega) + x_{i-1}, 0)$$

$$L = \max(W_n + Z_n(\omega) + \Sigma x_i - d, 0)$$

$$\min\{\overbrace{\sum_{i=1}^{n} C_i^w * E_\xi[W_i]}^{\text{Cost of Waiting}} + \overbrace{\sum_{i=1}^{n} C^s * E_\xi[S_i]}^{\text{Cost of Idling}} + \overbrace{C^L * E_\xi[L]}^{\text{Cost of Overtime}}\}$$

Stochastic Linear Program

$$\min\{E_{\xi}[\sum_{i=2}^{n} c_i^w w_i + \sum_{i=2}^{n} c^s s_i + c^L l]\}$$

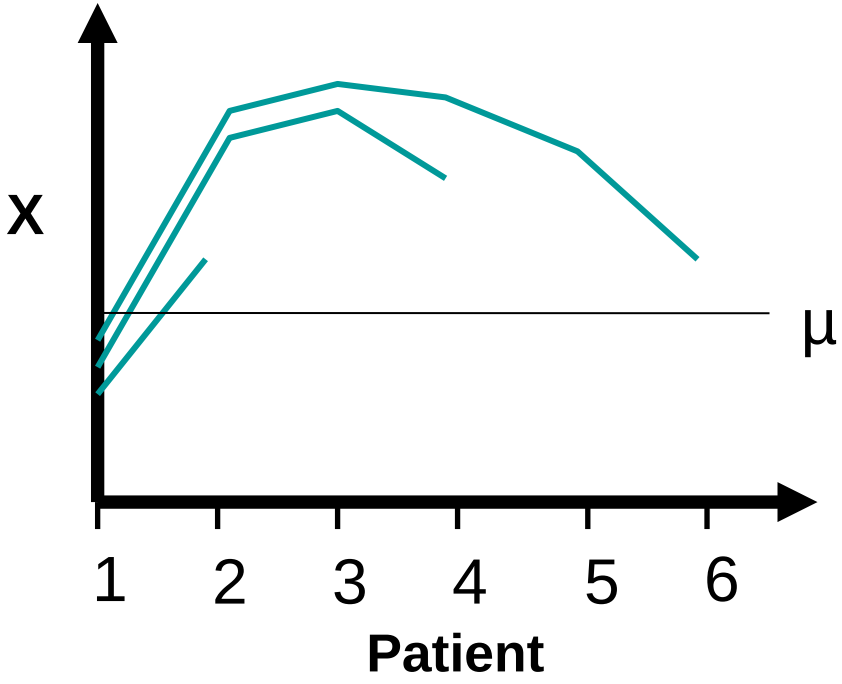
s.t.

$$\begin{array}{llll}
w_2 & & -s_2 & = Z_1(\omega) - x_1 \\
-w_2 + w_3 & & -s_3 & = Z_2(\omega) - x_2 \\
& \ddots & \ddots & \\
& -w_n & -s_n + l - g & = Z_n(\omega) - d + \sum_{j=1}^{n-1} x_i
\end{array}$$

$$x_i \geq 0, w_i \geq 0, s_i \geq 0, i = 1, \ldots, n, \quad l, g \geq 0$$

Example

- Comparison of surgery allocations for n=3, 5, 7 with i.i.d. distributions with U(1,2):



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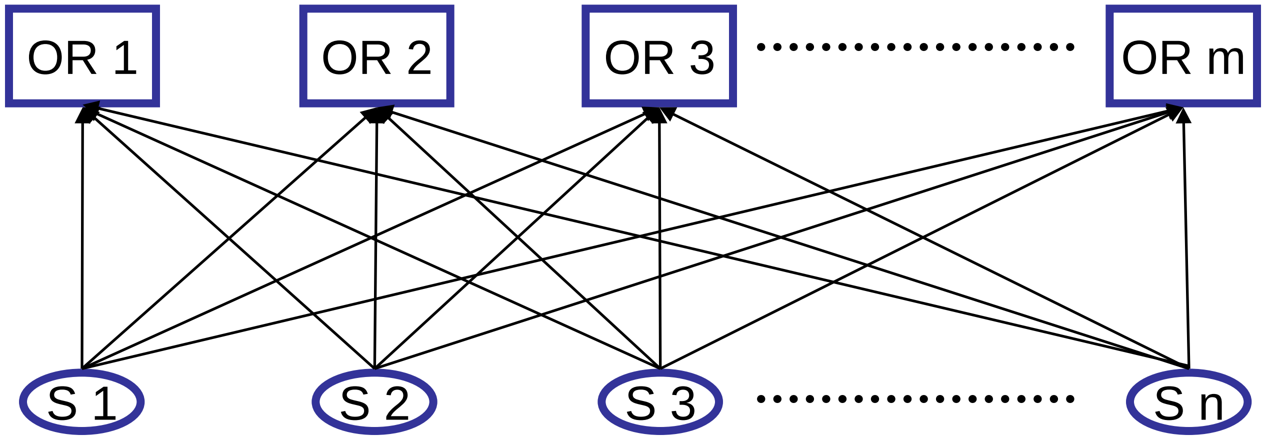
Problem Description

- What are the first and second stage variables?
- What are the first stage costs, c?
- What is the technology matrix, T?
- What is the recourse matrix, W?
- What is the the rhs, h?

Denton, B.T. and Gupta D., 2003, “A Sequential Bounding Approach for Optimal Appointment Scheduling,” *IIE Transactions*, 35, 1003-1016

Multi-OR Surgery Allocation

Multi-Operating Room Scheduling



Decisions:

- How many operating rooms (ORs) to open?
- Which OR to schedule each surgery in?

Objective:

- Cost of operating rooms opened
- Overtime costs for operating rooms

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Extensible Bin Packing

$$x_j = \begin{cases} 1 \ if\ OR\ \ j\ open \\ 0\ \ if\ \ OR\ \ j\ closed \end{cases} \qquad y_{ij} = \begin{cases} 1\ if\ Surgery\ i\ assigned\ to\ OR\ \ j \\ 0\ \ \ Otherwise \end{cases}$$

$$Z = \min\{\sum_{j=1}^{m} c^f x_j + c^v o_j\}$$

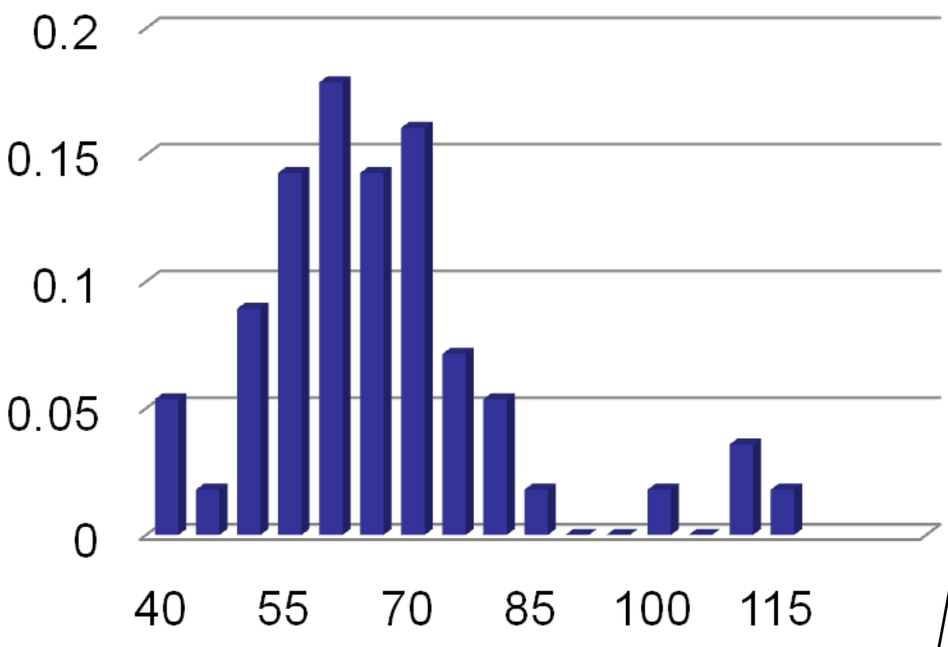
$$s.t. \qquad y_{ij} \le x_j \quad \forall(i,j)$$

$$\sum_{j=1}^{m} y_{ij} = 1 \ \ \forall(i)$$

$$\sum_{i=1}^{n} \mu_i y_{ij} - o_j \le d_j x_j \quad \forall(i,j)$$

$$y_{ij}, x_j \in \{0,1\}, \quad o_j \ge 0$$

Two-Stage Stochastic MIP



$$\min\{\sum_{j=1}^{m} c^f x_j + c^v E_\xi[o_j(\omega)]\}$$

$$s.t. \quad y_{ij} \leq x_j \quad \forall (i,j)$$

$$\sum_{j=1}^{m} y_{ij} = 1 \quad \forall (i)$$

$$\sum_{i=1}^{n} z_i(\omega) y_{ij} - o_j(\omega) \leq d_j x_j \quad \forall (i,j,\omega)$$

$$y_{ij}, x_j \in \{0,1\}, \quad o_j(\omega) \geq 0, \forall \omega$$

Questions

- What special properties does this problem have?

Denton, B.T. , Miller, A.J., Balasubramanian, H.J., Huschka, T.R., 2010, "Optimal Allocation of Surgery Blocks to Operating Rooms Under Uncertainty", *Operations Research*, 58(4), 802-816