Stochastic Programming
Lecture 5

Dr. Brian Denton

NCSU

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Today's Class

1. Feasibility sets for 2 Stage Recourse Problems
2. Properties of the Recourse Function
3. Types of Recourse
4. Types of Recourse

Outline

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Feasibility sets for 2 Stage Recourse Problems

- Much of todays lecture is based on Chapter 3, Section 3.1, of B & L
- The following is also a good reference: Wets, R.J-B. 1974, “Stochastic Programs with Fixed Recourse: The Equivalent Deterministic Program”, SIAM Review, 16(3), p. 309-339.

Feasibility sets for 2 Stage Recourse Problems

For given fixed x and ξ, the value, $Q(x,\xi)$, of the second-stage is given by

$$Q(x,\xi) = \min_y \{q(\omega)^T y | W(\omega) y = h(\omega) - T(\omega) x, y \geq 0\}$$

When $Q(x,\xi)$ is unbounded below define $Q(x,\xi) = -\infty$; when $Q(x,\xi)$ is infeasible define $Q(x,\xi) = +\infty$

- Let $Q(x) = E_\xi[Q(x,\xi)]$; $K_1 = \{x | Ax = b, x \geq 0\}$ is the set determined by the fixed constraints;
- Let $K_2 = \{x | Q(x) < \infty\}$ be the second-stage feasibility set. The DEP can be redefined as

$$\begin{aligned} &\min \ z(x) = c^T x + Q(x) \\ &\text{s.t.} \\ &\ \ x \in K_1 \cap K_2 \end{aligned}$$

Feasibility sets for 2 Stage Recourse Problems

$K_2(\xi) = \{x | Q(x, \xi) < +\infty\}$ is referred to as the *elementary feasibility set*. The *possibility interpretation* is defined as:

$$\begin{aligned} K_2^P &= \{x | \; \forall \xi \in \Xi, \; \exists y \geq 0 \text{ such that } Wy = h(\omega) - T(\omega)x\} \\ &= \cap_{\xi \in \Xi} K_2(\xi) \end{aligned}$$

Q. Are K_2 and K_2^P the same?

Feasibility sets for 2 Stage Recourse Problems

Theorem 1 (B&L, Ch 3).

a. For each ξ, the elementary feasibility set is a closed convex polyhedron, hence the set K_2^P is closed and convex.
b. When Ξ is finite, then K_2^P is also polyhedral and coincides with K_2.

Proof:

(a) K_2^P is the intersection of polyhedrons.

(b) Show that $K_2^P = K_2$.
Part 1: If $x \in K_2$ then $Q(x) < \infty$. Since $Q(x) = \sum_{\forall\omega} p(\omega)Q(x, \xi(\omega))$ it follows that $x \in K_2(\xi), \forall \xi \in \Xi$.
Part 2: If $x \in K_2^P$ then $Q(x, \xi) < \infty, \forall \xi \in \Xi$ which implies $Q(x) = \sum_{\forall\omega} p(\omega)Q(x, \xi(\omega)) < \infty$

Feasibility sets for 2 Stage Recourse Problems

Proposition 2 (B&L, Ch 3).

If ξ has finite second moments, then

$$P(\omega | Q(x, \xi(\omega)) < \infty) = 1 \textit{ implies } \mathcal{Q}(x) < \infty$$

Proof Sketch:
Solving the LP, $Q(x, \xi)$, yields some basis, B, such that $y_B^* = B^{-1}(h(\omega) - T(\omega)x)$ and $y_N^* = 0$. Therefore

$$Q(x, \xi) = q_B(\omega) B^{-1}(h(\omega) - T(\omega)x)$$

$Q(x, \xi)$ is bounded above w.p. 1 (by assumption). Thus ξ having bounded second moments is sufficient for $\mathcal{Q}(x) < \infty$ (for the region of Ξ such that basis B is optimal). Wets (1974) generalizes this argument to multiple optimal bases.

Feasibility sets for 2 Stage Recourse Problems

Theorem 3 (B&L, Ch 3).

For a stochastic program with fixed recourse where ξ has finite second moments, the sets K_2 and K_2^P coincide.

Proof:

Part 1:

If $x \in K_2^P$ then $Q(x,\xi) < \infty$ w.p. 1. Therefore $Q(x) < \infty$ and $x \in K_2$.

Part 2:

If $x \in K_2$ it follows that $\{\xi \mid Q(x,\xi) < \infty\}$ is a set of measure 1. $Q(x,\xi) < \infty$ implies that $h(\omega) - T(\omega)x \in posW$. Given some set Σ of measure one, $\{\xi \in \Sigma \mid Q(x,\xi) < \infty\}$ is a closed subset of Σ of measure one. In particular, the set $\{\xi \in \Xi \mid Q(x,\xi) < \infty\}$ is also a closed subset of Ξ of measure one. By definition of Ξ it follows $\{\xi \in \Xi \mid Q(x,\xi) < \infty\} = \Xi$, and therefore $x \in K_2^P$.

Feasibility sets for 2 Stage Recourse Problems

To summarize....

- If Ξ is finite then $K_2 = K_2^p$ and K_2 is polyhedral
- If Ξ is not finite but ξ has finite second moments then $K_2 = K_2^p$ and K_2 is closed and convex

Feasibility sets for 2 Stage Recourse Problems

Theorem 4 (B&L, Ch 3).

When W is fixed and ξ has finite second moments:

- *(a) K_2 is closed and convex*
- *(b) If T is fixed, K_2 is polyhedral.*
- *(c) Let Ξ_T be the support of the distribution of T. If $h(\omega)$ and $T(\omega)$ are independent and Ξ_T is polyhedral, then K_2 is polyhedral.*

Proof: Part (a) follows immediately from prior theorems (why?)
Part (b) (basic idea)....
If T is fixed then $x \in K_2$ iff $h(\omega) - Tx \in posW$ for all $\xi \in \Xi_h$. Use the *polar matrix*, W^*, to construct a polyhedral representation for the constraint $h(\xi) - Tx \in posW$.
See Wets (1974) for a proof of part (c).

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Properties of the Recourse Function

Theorem 5 (B&L, Ch 3).

For a stochastic program with fixed recourse, $Q(x, \xi)$ is

- *a piecewise linear convex function in (h, T);*
- *a piecewise linear concave function in q;*
- *a piecewise linear convex function in x for all $x \in K = K_1 \cap K_2$.*

Proof: This follows from basic properties of LPs....

Properties of the Recourse Function

Theorem 6 (B&L, Ch 3).

For a stochastic program with fixed recourse where ξ has finite second moments,

- *(a) $Q(x)$ is a Lipschitzian convex function and is finite on K_2;*
- *(b) When Ξ is finite, $Q(x)$ is piecewise linear;*
- *(c) If $F(\xi)$ is an absolutely continuous distribution, $Q(x)$ is differentiable on K_2.*

Proof: (a) and (b) follow from Theorem 5. See Wets (1972) or Kall (1976) for conditions under which $Q(x)$ is differentiable.

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Key points from section 3.1:

- K_2 and K_2^P are the same if W is fixed and either (a) ξ has finite second moments or (b) Ξ is finite. For many practical problems the latter is a reasonable assumption (why?).
- Given the above assumptions, K_2 is a closed convex polyhedron and $Q(x)$ is convex on K_2

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Types of Recourse

Special cases of 2nd stage feasibility for a given first stage decision x:

- *Relatively Complete Recourse*: $K_1 \subset K_2$
- *Complete Recourse*: This occurs when the $pos(W) = \Re^{m_2}$, i.e., W contains a positive linear basis of $\Re^{m_2}$.
- *Simple Recourse:* This is a special type of complete recourse in which $W = [I \mid -I]$ and y is divided correspondingly as (y^+, y^-) and $q = (q^+, q^-)$