Stochastic Programming
Lecture 6

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Today's Class

1. Probabilistic or Chance Constraints
2. Stochastic Integer Programming
3. Multi-Stage Stochastic Linear Programs (MSLP)
4. Summary

Outline

Probabilistic or Chance Constraints

The form of probabilistic (chance) constraints:

$$\mathrm{P}\{A^i(\omega)x \geq h^i(\omega)\} \geq \alpha^i$$

where $\alpha^i \in [0, 1]$ denotes the confidence level.

The feasible sets are

$$K_1^i(\alpha^i) = \{x | P\{A^i(\omega)x \geq h^i(\omega)\} \geq \alpha^i\}$$

and $K_1 = \cap_i K_1^i(\alpha^i)$

Note: when $\alpha^i = 0$ or 1, the chance constraints reduce to deterministic constraints.

Probabilistic or Chance Constraints

- The feasible region defined by chance constraints is <u>not necessarily convex or connected</u>

Probabilistic or Chance Constraints

- The feasible region defined by chance constraints is not necessarily convex or connected
- For the special case of a single linear constraint

$$P\{Ax \geq h(\omega)\} = F(Ax)$$

where $F(\cdot)$ is the cdf of $h(\omega)$ and

$$K_1(\alpha) = \{x \mid F(Ax) \geq \alpha\} = \{x \mid Ax \geq F^{-1}(\alpha)\}$$

Probabilistic or Chance Constraints

In the joint constraint case there is class of probability distributions for which $K_1(\alpha)$ is provably convex.

Definition: A probability measure, P, is *quasi-concave*, if for any convex measurable sets U and V and $0 \leq \lambda \leq 1$:

$$P((1-\lambda)U + \lambda V) \geq \min\{P(U), P(V)\}$$

This class includes normal, beta, and Dirichlet distributions.

Definition: A probability measure is *logarithmically-concave* if:

$$P((1-\lambda)U + \lambda V) \geq P(U)^{\lambda} P(V)^{1-\lambda})$$

Probabilistic or Chance Constraints

Theorem 16 (B&L Ch 3).

Suppose A is fixed and the components $h_i, i = 1, ..., m_1$, of vector h are independent random variables with logarithmically concave probability measures, $P_i(\cdot)$, and distribution functions, $F_i(\cdot)$, then $K_1(\alpha)$ is convex.

Proof:

$P(Ax \geq h) = \prod_{i=1}^{m_1} P_i(A_i.x \geq h_i) = \prod_{i=1}^{m_1} F_i(A_i.x)$ and $K_1(\alpha) = \{x \mid F_i(A_i.x) \geq \alpha)\}$. Taking logarithms (a monotonically increasing function), it follows that $K_1(\alpha) = \{x \mid \sum_{i=1}^{m_1} ln(F_i(A_i.x)) \geq ln(\alpha)\}$. Because

$$F_i(A_i.(\lambda x^1 + (1-\lambda)x^2) \geq F_i(A_i.x^1)^{\lambda} F_i(A_i.x^2)^{1-\lambda},$$

$ln(F_i(A_i.x))$ is a concave function and it follows $K_1(\alpha)$ is convex.

Outline

Stochastic Integer Programming

Definition: Two Stage Stochastic Integer Programs:

$$\begin{aligned} &\min = c^T x + Q(x) \\ &\text{s.t. :} \\ &\quad Ax = b, x \in X \end{aligned}$$

$$Q(x) = \mathrm{E}_{\xi}[\min_{y \in Y}\{q(\omega)^T y \mid Wy = h(\omega) - T(\omega)x\}]$$

where X and/or Y contains some integrality restrictions on x and/or y.

How do integrality constraints affect $Q(x)$ and K_2?

Stochastic Integer Programming

Some important distinctions between 2-SLP and 2-SIP with integer recourse from Birge and Louveaux, Section 3.3....

Proposition 20 (B&L Ch 3).

The expected recourse function $Q(x)$ of an integer program is in general nonconvex and discontinuous.

Prosposition 22 (B&L Ch 3).

The second-stage feasibility set $K_2(\xi)$ is in general nonconvex.

In general, problems with integer recourse are very hard to solve.

Simple Integer Recourse

Integrality restrictions can make things a lot harder

- Integrality restrictions on x do not interfere with properties of $Q(x)$
- Integrality restrictions on y are worse because many of the useful properties of the recourse function are lost
- Following is the simple integer recourse problem...

$$\min cx + E_{\xi}[\min\{q^{+}y^{+} + q^{-}y^{-} \mid y^{+} \geq \xi - Tx,\ y^{-} \geq Tx - \xi,\ y^{+}, y^{-} \in Z_{+}^{m}\}$$

s.t. :

$$Ax = b,\ x \in X$$

Outline

Multi-Stage Stochastic Linear Programs (MSLP)

2-SLP can be generalized to multiple stages (MSLP)

$$
\begin{aligned}
&\min c^1 x^1 + E_{\xi^2}[\min c^2(\omega)x^2(\omega^2) + \cdots + E_{\xi^H}[\min c^H(\omega)x^H(\omega^H)]\cdots] \\
&\text{s.t. :} \\
&\;\; W^1 x^1 = h^1 \\
&T^1(\omega)x^1 + W^2 x^2(\omega^2) = h^2(\omega) \\
&\vdots \\
&T^{H-1}(\omega)x^{H-1}(\omega^{H-1}) + W^H x^H(\omega^H) = h^H(\omega) \\
&x^1 \geq 0;\;\; x^t(\omega^t) \geq 0,\;\; t = 2, ..., H;
\end{aligned}
$$

where $c^1 \in \Re^{n_1}$ and $h^1 \in \Re^{m_1}$ are deterministic, $\xi^t(\omega) = (c^t(\omega), h^t(\omega), T_1^{t-1}(\omega), ..., T_{m_t}^{t-1})$ is a random N_t-vector defined on (Ω, Σ^t, P) (where $\Sigma^t \subset \Sigma^{t+1}$) for all $t = 2, ..., H$.

Multi-Stage Stochastic Linear Programs (MSLP)

Decisions x depend on the history up to time t, denoted by ω^t. The support of ξ^t is Ξ^t.

The MSLP can be expressed as a dynamic program with stages 1 to H, and states $x^t(\omega^t)$. The terminal condition is

$$
\begin{aligned}
&Q^H(x^{H-1}, \xi^H(\omega)) = \min c^H(\omega) x^H(\omega) \\
&\text{s.t. :} \\
&\;\; W^H x^H(\omega) = h^H(\omega) - T^{H-1}(\omega) x^{H-1} \\
&x^H(\omega) \geq 0
\end{aligned}
$$

Let $Q^{t+1}(x^t) = E_{\xi^{t+1}}[Q^{t+1}(x^t, \xi^{t+1}(\omega))]$ for all t. The MSLP can be expressed as a dynamic program with stages 1 to H, and states $x^t(\omega^t)$.

Multi-Stage Stochastic Linear Programs (MSLP)

The period t recourse problem is

$$
\begin{aligned}
&Q^t(x^{t-1}, \xi^t(\omega)) = \min c^t(\omega)x^H(\omega) + Q^{t+1}(x^t)\\
&\text{s.t. :}\\
&\quad W^t x^t(\omega) = h^t(\omega) - T^{t-1}(\omega)x^{t-1}\\
&x^t(\omega) \geq 0
\end{aligned}
$$

The complete deterministic equivalent problem is:

$$
\begin{aligned}
&\min c^1 x^1 + Q^2(x^1)\\
&\text{s.t. :}\\
&\quad W^1 x^1 = h^1\\
&x^1 \geq 0
\end{aligned}
$$

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Summary

Things we learned today (in plain english)...

- Probabilistic constraints can be hard to deal with because constraints induced on the first stage are not necessarily convex or even connected
- Integer variables on the first stage make the problem harder, and integer constraints on the second stage make the problem <u>much harder</u> because nice properties of the recourse function (like convexity) are lost
- Multi-stage stochastic linear programs are hard to solve because they grow very large as the number of stages grows (the curse of dimensionality)