Stochastic Programming: Lecture 3

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August 31, 2010

Today's Class

1. Review of Linear Programming
2. Review of Probability Spaces
3. Decisions and Stages
4. Multi-Stage Example: Financial Planning and Control

Outline

1. Review of Linear Programming
2. Review of Probability Spaces
3. Decisions and Stages
4. Multi-Stage Example: Financial Planning and Control

Review of Linear Programming

Since stochastic linear programs are extensions of linear programs (LPs) we will draw heavily on properties of LPs, algorithms for solving large-scale LPs, duality, and other important topics you covered in (ISE/OR505). Some important things to know....
Deterministic LP in standard form:

$$\min\{cx | Ax = b, x \geq 0\}$$

where $x, c \in \Re^n$, $b \in \Re^m$, $A \in \Re^{m \times n}$.

- Polyhedral theory
- LP structural properties
- Simplex method
- Duality
- Branch-and-bound

Outline

Review of Probability Spaces

Since we are going to be talking a lot about random variables, let's formalize the concepts of random variables...

The triplet $(\Omega, \mathcal{A}, P)$ defines a *probability space*:

- Ω is the set of all outcomes (indexed by ω)
- $\mathcal{A}$ is the set of all subsets of Ω, called *events*
- P is a *probability measure*

Probability spaces must satisfy certain properties:

Axiom 1. $P\{\Omega\} = 1$.

Axiom 2. $P\{A\} \geq 0$ *for any* $A \in \mathcal{A}$.

Axiom 3. *For every countable sequence of mutually disjoint events* $\{A_i\}_{i=1}^{\infty}$, *we have*

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}.$$

Review of Probability Spaces

Definition

A random variable is a measurable function from a probability space $(\Omega, \mathcal{A}, P)$ to the set of real numbers.

Example:

Let $\Omega = \{\omega_1, \omega_2\}$ and $P\{\omega_1\} = P\{\omega_2\} = 0.5$. Then the function

$$\xi(\omega) = \begin{cases} -1, & \text{if } \omega = \omega_1 \\ 1, & \text{if } \omega = \omega_2 \end{cases}$$

is a random variable.

Review of Probability Spaces

For a particular random variable, ξ, define a *cumulative distribution function* as $F_\xi(x) = P(\xi \leq x)$ (or as $F_\xi(x) = P(\omega \,|\, \xi \leq x\})$)

- Discrete random variable are described by finite or countable outcomes, $\xi(\omega^k), k \in K$
- Continuous random variables are described by a *probability density function* (pdf) where:

$$P(a \leq \xi \leq b) = \int_a^b f(\xi) d\xi$$

Outline

Decisions and Stages

Recourse problems have *recourse actions* that can be taken after the uncertain parameters are disclosed. Decisions are divided into 2 groups:

- *first stage decisions* that are made before the uncertain parameters are disclosed
- *second stage decisions* that are made after the uncertain paraemters are disclosed

The sequence of events can be summarized as:

$$x \rightarrow \xi(\omega) \rightarrow y(\omega, x)$$

This can be generalized to multiple stages....

Outline

Multi-Stage Example: Financial Planning and Control

Example (College Savings Plan): Suppose you want to save for a child's college education 15 years from now. You currently have $\$b$ to invest in stocks ($i = 1$) and bonds ($i = 2$) and you plan to rebalance your portfolio every 5 years. In each 5-year period there are two possible outcomes:

- Stocks return 1.25 and bonds return 1.14 (good scenario)
- Stocks return 1.06 and bonds return 1.12 (bad scenario)

Let s_t index the two possible outcomes in each period. In total there are 8 possible scenarios $\{LLL, LLH, LHL, ..., HHH\}$.

Assume each scenario is equally likely, i.e.,
$p(s_1, s_2, s_3) = 0.125, \forall(s_1, s_2, s_3)$

Model Formulation

Let $\xi(i, t, s_t)$ denote the return for investment i, in period t, and outcome, s_t.

Decision Variables:

- $x(i, t, s_1, ..., s_{t-1})$ is the amount of investment i in period t given history $\{s_1, ..., s_{t-1}\}$.
- $w(s_1, s_2, s_3)$ and $y(s_1, s_2, s_3)$ are *shortage* and *surplus* variables at the end of the planning horizon.
- exceeding target G results in you getting an income of q% of the surplus, $y(s_1, s_2, s_3)$
- not meeting the goal requires borrowing at a cost of r% of the shortage, $w(s_1, s_2, s_3)$

Model Formulation

$$
\begin{aligned}
&\max \textstyle\sum_{s_3} \sum_{s_2} \sum_{s_1} p(s_1, s_2, s_3)(-rw(s_1, s_2, s_3) + qy(s_1, s_2, s_3)) \\
&\text{s.t. :} \\
&\quad \textstyle\sum_i x(i, 1) = b \\
&\quad \textstyle\sum_i -\xi(i, 1, s_1)x(i, 2) + \sum_i x(i, 1, s_1) = 0 \\
&\quad \textstyle\sum_i -\xi(i, 2, s_1, s_2)x(i, 2, s_1) + \sum_i x(i, 2, s_1, s_2) = 0 \\
&\quad \textstyle\sum_i \xi(i, 3, s_1, s_2, s_3)x(i, 3, s_1, s_2) - y(s_1, s_2, s_3) + w(s_1, s_2, s_3) = G \\
&\quad x(i, t, s_1, ..., s_{t-1}) \geq 0, y(s_1, s_2, s_3) \geq 0, w(s_1, s_2, s_3) \geq 0, \forall i, t, s_1, s_2, s_3
\end{aligned}
$$

Model Formulation

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\begin{aligned}
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&\text{s.t. :}\\
&\quad \textstyle\sum_i x(i,1)=b\\
&\quad \textstyle\sum_i -\xi(i,1,s_1)x(i,2)+\sum_i x(i,1,s_1)=0\\
&\quad \textstyle\sum_i -\xi(i,2,s_1,s_2)x(i,2,s_1)+\sum_i x(i,2,s_1,s_2)=0\\
&\quad \textstyle\sum_i \xi(i,3,s_1,s_2,s_3)x(i,3,s_1,s_2)-y(s_1,s_2,s_3)+w(s_1,s_2,s_3)=G\\
&\quad x(i,t,s_1,...,s_{t-1})\geq 0, y(s_1,s_2,s_3)\geq 0, w(s_1,s_2,s_3)\geq 0, \forall i,t,s_1,s_2,s_3
\end{aligned}
$$

You can solve this deterministic equivalent problem (DEP) for the optimal action at each period for each possible history of outcomes.

Model Formulation

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\end{aligned}
$$

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What happens as the number of periods increases?

Probability Constrained Example

The deterministic setcovering problem is:

$$\min\{\sum_{j=1}^{n} c_j x_j \mid \sum_{j \in N_i} x_j \geq 1, \forall i, x_j \in \{0,1\}, \forall j\}$$

In the probabilistic version of set covering assume that selected sets may be "unavailable" with probability q and $P(\text{at least one set covers item } j) \geq \alpha$ is required. Thus the probability an item is "covered" is:

$$1 - q^{\sum_{j \in N_i} x_j} \geq \alpha, \forall i$$

This can be converted to a deterministic linear constraint (how?)

Relationships to Other Decision Making Models

How does stochastic programming relate to other methods for decision making under uncertainty?

Relationships to Other Decision Making Models

How does stochastic programming relate to other methods for decision making under uncertainty?

- Decision analysis
- Dynamic programming and Markov decision processes
- Stochastic control