ISE789 - Stochastic Programming

Assignment # 2 - Due September 30, 2010

Note: When answering assignment questions be sure to show all of your work. Points will be allocated for each step of the solution process.

Question 1 (8 Points): Consider the following random linear program:

$$\begin{array}{ll} Q(x,\xi) = & \min y_1(\omega) + y_2(\omega), \\ \text{s.t.}: & y_1(\omega) - y_2(\omega) = x - \xi, \\ & y_1(\omega), y_2(\omega) \geq 0 \end{array}$$

Where $\xi \sim U(0,1)$ and ω indexes outcomes of ξ and $x \geq 0$. Let $y_1^*(\omega)$ and $y_2^*(\omega)$ be the optimal solution given some x, and some outcome, ω, of random variable ξ.

a. Write $y_1(\omega)^*$, $y_2(\omega)^*$, $Q(x,\xi)$, as a function of x and ξ.

b. Solve the *distribution problem* to find, $F_Q(\cdot)$, the cumulative distribution function for $Q(x,\xi)$.

c. Use $F_Q(\cdot)$ to find $x^* = \arg\min E[Q(x,\xi)]$.

Answer: The purpose of this question was to illustrate the difficulty of solving the *distribution problem* in stochastic programming.

Part (a): The optimal solution can be written as;

$$y_1(\omega)^* = (x-\xi)^+, \quad y_2(\omega)^* = (\xi - x)^+, \qquad Q(x,\xi) = (x-\xi)^+ + (\xi - x)^+$$

Part (b): The *distribution problem* involves finding the cumulative distribution function (cdf) for the optimal solution to the random LP, $Q(x,\xi)$. We define the cdf as

$$\begin{aligned} F_Q(a) &= P(Q(x,\xi) \leq a) && (1) \\ &= P((x-\xi)^+ + (\xi - x)^+ \leq a) && (2) \\ &= P(x - a \leq \xi \leq x + a). && (3) \\ & && (4) \end{aligned}$$

Given that $x^* \in [0,1]$ we can split the cdf into two parts conditional on x as follows:

If $x \leq 1/2$

$$F_Q(a) = \begin{cases} 0 & \text{if } a < 0 \\ 2a & 0 \leq a \leq x \\ x + a & x \leq a \leq 1 - x \\ 1 & a > 1 - x \end{cases}$$

If $x \geq 1/2$

$$F_Q(a) = \begin{cases} 0 & \text{if } a < 0 \\ 2a & 0 \leq a \leq 1 - x \\ x + a & 1 - x \leq a \leq x \\ 1 & a > x \end{cases}$$

Part (c): It follows that the expectation can be written terms of $F_Q(Q)$ as follows:

For $x \leq 1/2$

$$E[Q] = 2\int_0^x QdF_Q(Q) + \int_x^{1-x} QdF_Q(Q)) = \frac{(1-x)^2}{2} + \frac{x^2}{2}$$

For $x \geq 1/2$

$$E[Q] = 2\int_0^{1-x} QdF_Q(Q)) + \int_{1-x}^{x} QdF_Q(Q)) = \frac{(1-x)^2}{2} + \frac{x^2}{2}$$

Thus E[Q] is the same on both regions of x, and it is continuous and differentiable. Taking the second derivative the minimum is reached at $x^* = 1/2$.

Question 2 (3 Points): For a 2-stage stochastic linear program with fixed recourse prove that $Q(x, \xi)$ is concave in the second stage cost coefficients, q.

Answer:

Given two cost vectors q_1 and q_2 let:

$$f(q_1) = \min\{q_1 y \mid Wy = h - Tx, y \geq 0\} = q_1 y_1^*$$

$$f(q_2) = \min\{q_2 y \mid Wy = h - Tx, y \geq 0\} = q_2 y_2^*$$

$$f(q_\lambda) = \min\{(\lambda q_1 + (1-\lambda) q_2) y \mid Wy = h - Tx\} = q_\lambda y_\lambda^*$$

for some $\lambda \in [0,1]$. For $f(\cdot)$ to be concave in q the following is a necessary and sufficient condition:

$$f(q_\lambda) \geq \lambda f(q_1) + (1-\lambda)f(q_2), \forall \lambda \in [0,1].$$

From optimality conditions we have that $q_1 y_1^* \leq q_1 y_\lambda^*$ and $q_2 y_2^* \leq q_2 y_\lambda^*$. It follows that:

$$\lambda q_1 y_\lambda^* + (1-\lambda) q_2 y_\lambda^* = f(q_\lambda) \geq \lambda q_1 y_1^* + (1-\lambda) q_2 y_2^* = \lambda f(q_1) + (1-\lambda) f(q_2)$$

and therefore $f(\cdot)$ is concave in q.

Question 3 (3 Points): Prove that the intersection of two convex sets is a convex set.

Answer: Given two sets A and B if $x^1, x^2 \in A \cap B$ then for $\lambda \in [0,1]$ it follows that $\lambda x^1 + (1-\lambda)x^2 \in A$ and $\lambda x^1 + (1-\lambda)x^2 \in B$ since A and B are both convex. Therefore $A \cap B$ is convex since any convex combination of x^1 and x^2 lies in $A \cap B$.

Question 4 (5 Points): Let the second stage of a stochastic program be

$$\begin{array}{rl} & \min 2y_1(\omega) + y_2(\omega) \\ \text{s.t. :} & y_1(\omega) - y_2(\omega) \leq 2 - \xi x_1, \\ & y_2(\omega) \leq x_2, \\ & y_1(\omega), y_2(\omega) \geq 0 \end{array}$$

Find $K_2(\xi)$ and K_2 for:

a. $\xi \sim U(0,1)$

b. $\xi \sim Poisson(\lambda), \lambda > 0$

What properties do you expect for K_2?

Answer:

For part (a) we have

$$K_2(\xi) = \{x_1, x_2 \mid x_2 \geq 0, x_2 - \xi x_1 \geq -2\}\}$$

It follows that

$$K_2 = \cap_\xi K_2(\xi) = \{x_1, x_2 \mid x_2 \geq 0, x_2 - x_1 \geq -2\}$$

Given that the stochastic program has fixed recourse and ξ has finite second moments, from Theorem 3, p. 88, of Birge and Louveaux, it follows that K_2 and K_2^P coincide.

For part (b) we have the same $K_2(\xi)$ as above. However, in this case the support is unbounded. As $\xi \to \infty$ the following must hold

$$x_2 - \xi x_1 \geq -2$$

This implies that $x_1 \leq 0$. Therefore K_2 can be written as:

$$K_2 = \{x_1, x_2 \mid x_2 \geq 0, x_1 \leq 0\}$$

Given that the stochastic program has fixed recourse and ξ has finite second moments, from Theorem 3, p. 88, of Birge and Louveaux, it follows that K_2 and K_2^P coincide.

Question 5 (6 Points): Consider the following second stage recourse problem:

$$Q(x, \xi) = \min\{\xi y | y \geq \xi, y \geq x\} \tag{5}$$

Assume $x \geq 0$. Let ξ have probability density function $f(\xi) = 2/\xi^3, \xi \geq 1$.

Show that $K_2^P \neq K_2$. What sufficient condition for coincidence of K_2^P and K_2 does this distribution violate?

Answer: For any particular ξ $Q(x, \xi)$ has optimal solution $y = \min(x, \xi)$. Therefore $K_2(\xi) = \Re$. However, $E[Q(x, \xi)] = \infty, \forall x \in \Re$, i.e., the recourse function is not bounded and K_2 is empty. Therefore K_2 and K_2^P do not coincide. The sufficient condition that is violated is the condition of finite second moments since $E[\xi^2] = \infty$.

Question 6 (8 Points): Consider the following optimization problem:

$$\min\{x_1 + 2x_2 + E_\xi[\max\{(\xi_1 - x_1), 0\} + \max\{\max\{(\xi_1 - x_1), 0\} + \xi_2 - x_2, 0\} + \tag{6}$$

$$\max\{(x_1 - \xi_1), 0\} + \max\{(x_2 - \xi_2), 0\}] | x_1 + x_2 \geq 1, x_1, x_2 \geq 0\} \tag{7}$$

where ξ_1, ξ_2 are random variables.

a. Show that it can be formulated as a two stage stochastic linear program with fixed recourse. Use the notation of chapter 3, Birge and Louveaux, i.e., write c, q, T, W, h.

b. Assuming ξ_1 and ξ_2 have a discrete distribution with outcomes $\{1, 1.5, 2, 2.5, 3\}$ occurring with equal probability, compute the Value of the Stochastic Solution (VSS) and the expected value of perfect information (EVPI)?

c. Does the formulation from part (a) correspond to any of the following: complete recourse, relatively complete recourse, or simple recourse?

Answer:

Part (a): The above optimization problem can be rewritten as:

$$\begin{array}{ll} \min & x_1 + 2x_2 + E_\xi[Q(x,\xi)] \\ \text{s.t.} & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

where $\xi = (\xi_1, \xi_2)$ and

$$\begin{array}{rl} Q(x,\xi) = & \min y_1 + y_2 + y_3 + y_4 \\ & \text{s.t. } y_1 \geq \xi_1 - x_1 \\ & y_2 - y_1 \geq \xi_2 - x_2 \\ & y_3 \geq x_1 - \xi_1 \\ & y_4 \geq x_2 - \xi_2 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

Part (b): $VSS = EEV$ - RP and $EVPI = RP$ - WS. Therefore we need to compute RP, EV, EEV, and WS. RP is obtained by solving the deterministic equivalent problem (with $k = 1, ..., 25$ scenarios) using CPLEX:

$$\begin{array}{ll} RP = \min & x_1 + 2x_2 + \sum_{k=1}^{25}(1/25)(y_1^k + y_2^k + y_3^k + y_4^k) \\ \text{s.t.} & x_1 + x_2 \geq 1 \\ & y_1^k + x_1 \geq \xi_1^k, \ \forall k \\ & y_2^k - y_1^k + x_2 \geq \xi_2^k \ \forall k \\ & y_3^k - x_1 \geq -\xi_1^k \ \forall k \\ & y_4^k - x_2 \geq -\xi_2^k \ \forall k \\ & y_1^k, y_2^k, y_3^k, y_4^k, x_1, x_2 \geq 0 \ \forall k \end{array}$$

which yields $z^* = 4.8$. EV is obtained by setting $\xi_1 = 2, \xi_2 = 2$ and solving:

$$
\begin{array}{ll}
EV = \min & x_1 + 2x_2 + y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} & x_1 + x_2 \geq 1 \\
 & y_1 + x_1 \geq 2 \\
 & y_2 - y_1 + x_2 \geq 2 \\
 & y_3 - x_1 \geq -2 \\
 & y_4 - x_2 \geq -2 \\
 & y_1, y_2, y_3, y_4 \geq 0
\end{array}
$$

which has optimal solution $x_1^* = 1, x_2^* = 0$. EEV is obtained by fixing $x_1 = 1, x_2 = 0$ in RP and solving:

$$
\begin{array}{ll}
EEV = \min & (1) + 2(0) + \sum_{k=1}^{25}(1/25)(y_1^k + y_2^k + y_3^k + y_4^k) \\
\text{s.t.} & y_1^k \geq \xi_1^k - 1 \ \forall k \\
 & y_2^k - y_1^k \geq \xi_2^k \ \forall k \\
 & y_3^k \geq -\xi_1^k + 1 \ \forall k \\
 & y_4^k \geq -\xi_2^k \ \forall k \\
 & y_1^k, y_2^k, y_3^k, y_4^k, x_1, x_2 \geq 0 \ \forall k
\end{array}
$$

which yields $EEV = 4.9$ and $VSS = EEV - RP = 4.9 - 4.8 = 0.1$. WS is obtained by solving:

$$
\begin{array}{ll}
WS = \min & \sum_{k=1}^{25}(1/25)(2x_1^k + 3x_2^k + y_1^k + y_2^k + y_3^k + y_4^k) \\
\text{s.t.} & x_1^k + x_2^k \geq 1 \\
 & y_1^k + x_1^k \geq \xi_1^k \\
 & y_2^k - y_1^k + x_2^k \geq \xi_2^k \\
 & y_3^k - x_1^k \geq -\xi_1^k \\
 & y_4^k - x_2^k \geq -\xi_2^k \\
 & y_1^k, y_2^k, y_3^k, y_4^k, x_1^k, x_2^k \geq 0
\end{array}
$$

which has optimal solution $WS = 4$. Therefore $EVPI = 4.8 - 4 = 0.8$

Note: there were alternative solutions to RP leading to different values of VSS and EVPI.

Part (c): This problem has relatively complete recourse since $K_1 \subset K_2$, and complete recourse since $pos(W) = \Re^4$. This problem does not have simple recourse, i.e., the recourse matrix, W, is not separable.