Multi-Stage Stochastic Programming, CH. 7.1 B&L

"Decomposition and Partioning Methods for Multistage Stochastic Linear Programs" -Birge (1985) *Operations Research*

Sept. 3, 2010

What is Multi-Stage Stochastic Programming?

Motivation & Today's Goal

We want an approach similar to 2-stage L-shaped method and apply it to multistage stochastic programming problems

We will learn the basic concept and get a feel for the method

We could just use dynamic programming, but......why bad idea?

Notation

jumping right in

- H stages, $t = 1, 2, \ldots, H$
- J_t # of possible scenarios at stage t, $j_t = 1, 2, \ldots, J_t$ J_t finite
- $x_t^j =$ decision to make at stage t if scenario (scenario history really) j observed
- ξ_t =random outcome of event at (beginning of) stage t
- π dual variables associated with problem constraints, $Wx_t = h - Bx_{t-1}$
- ρ dual variables associated with feasibility conditions, $D_t^j x_t^j \geq d_t^j$(multiple ones possible)
- σ dual variables associated with optimality constaints, $E_t^j x_t^j \geq e_t^j$ (multiple ones possible)
- p_t^j =probability of event j in period t , $\Rightarrow \sum_{j \in J_t} p_t^j = 1$
- $a(j)$, $d(j)$, ancestor and descendant scenario(s) of scenario j

Timeline of Events

1. Stage 1 decision
2. Stage 2 outcome
3. Stage 2 decision (dependent on (1) and (2))
4. Stage 3 outcome
5. .....
6. Stage H outcome (dependent on sequence of outcomes)

Model Formulation

$$\min_{x^1} z = c_1 x_1 + E_{\xi_2}\left[\min_{x_2} c_2 x_2(\omega_2) + E_{\xi_3}\left[\min_{x_3} c_3 x_3(\omega_3) + \ldots E_{\xi_H}\left[\min_{x_H} c_H x_H(\omega_H)\right]\right]\right]$$

subject to:

$$\begin{array}{lllll}
W_1 x_1 & & & & = h_1 \\
B_1 x_1 & +W_2 x_2(\omega_2) & & & = h_2(\omega_2) \\
 & B_2 x_2 & +W_3 x_3(\omega_3) & & = h_3(\omega_3) \\
 & & \vdots & & \\
 & & \ldots & B_{H-1} x_{H-1} \quad +W_H x_H(\omega_H) & = h_H(\omega_H)
\end{array}$$

$$x \geq 0$$

Equivalent Deterministic Linear Program

For a three stage, 2 scenario problem

$$\min_{x_1} c_1 x_1 + \sum_{j=1}^{J_2} \left[p^j \left(c_2 x_2^j \right) \right] + \sum_{j=1}^{J_3} \left[p^j \left(c_3 x_3^j \right) \right]$$

subject to:

$$\begin{array}{ccccccccc}
W_1 x_1 & & & & & & & = & h_1 \\
\hline
B_1 x_1 & +W_2 x_2^1 & & & & & & = & h_2^1 \\
B_1 x_1 & & W_2 x_2^2 & & & & & = & h_2^2 \\
\hline
 & B_2 x_2^1 & & +W_3 x_3^1 & & & & = & h_3^1 \\
 & B_2 x_2^1 & & & W_3 x_3^2 & & & = & h_3^2 \\
 & & B_2 x_2^2 & & & W_3 x_3^3 & & = & h_3^3 \\
 & & B_2 x_2^2 & & & & W_3 x_3^4 & = & h_3^4
\end{array}$$

Tid Bits

- decision variable x_t^j, depends on past observations $[\omega_2, \omega_3, \ldots, \omega_t]$
- Recall L-shaped Method

Tid Bits

- decision variable x_t^j, depends on past observations $[\omega_2, \omega_3, \ldots, \omega_t]$
- Recall L-shaped Method
- L-shaped Method: (outer linearization)
  - solve stage one problem, send solution to second stage
  - If, for any scenario, the first stage solution is second-stage-infeasible, create feasibility condition and get new stage one solution
  - Repeat until a stage one solution is stage-two-feasible for all scenarios, then create optimality constaint, and resolve first stage problem
  - Repeat until optimality condition reached

Basic Idea of Nested Bender's Decomposition

L-Shaped Method for Multi-Stage SP

- Break down into many stage/scenario specific problems, of the form:

$$\min c_t x_t^j + \theta_t^j$$
$$s.t. \quad W x_t^j = h - B_{t-1} x_{t-t}^{a(j)}$$
$$D_t^{j,r} x_t^j \geq d_t^{j,r} \ldots \forall r$$
$$E_t^{j,s} x_t^j + \theta_t^j \geq e_t^{j,s} \ldots \forall s$$

- Forward Pass: Move from $t = 1$ to $t = H$, getting feasible solutions along the way (may require feasibility conditions and backtracking)
- Backwards: Move towards $t = 1$, making optimality constraints along the way
- Create optimality cuts of the expected future, while maintaining feasibility for all descendent problems

Steps

From Birge/Louveaux

- Step 0: Initialize $t = 1, j = 1, \theta = 0$ Go to step 1 (start forward pass)
- Step 1: Solve the t/j problem (using current $x_{t-1}^{a(j)}$ solution).
  - If infeasible, create feasibility condition for $t-1/a(j)$ problem, $t = t-1, j = a(j)$,return to step 1
  - If feasible, solve problem and remember dual solution, $j = j+1$
  - Continue until $d(j) \forall j \in J_t$ are feasible, then $t = t+1$, return to step 1
  - Continue until $t = H$, go to step 2 (begin backwards pass), set $j = 1$
- Step 2:
  - Check optimality for $t-1/j$ problem using dual info from $t/d(j)$ problems
  - If optimality condition created, add condition to $t-1/j$ problem, set $t = t-1, j = 1$, go to step 1 (begin forwards pass)
  - If no optimality condition reached, $j = j+1$, return to step 2
  - Continue until $j = J_t$, (no more optimality conditions for t), resolve all $t-1$ problems (remembering dual). If any infeasible, jump to step 1. If all feasible, set $t = t-1$, return to step 2
  - Continue until $t = 2$ and no optimality constraints are added ($x_1$optimal). If $t = 2$ and optimality cuts created, go to step 1

Feasibility and Optimality

Book's version

- "Feasibility" of a stage t solution x_t^j, needs to be checked only for $t+1$ descendant problems, $d(j_t)$
- feasibility condition like L-shaped method (v^+, v^-, w) etc.
    - for a dual basic solution, $\pi_t^j, \rho_t^j \geq 0$, s.t. $\pi_t^j W + \rho_t^j D_t^j \leq 0$ and $\pi_t^j \left(h_t^j - B_{t-1}^j x_{t-1}^{a(j)} \right) + \rho_t^j d_t^j > 0$
    - $D_{t-1}^{a(j)} = \pi_t^j B_{t-1}^j,\ d_{t-1}^j = \pi_t^j h_t^j + \rho_t^j d_t^j$
- optimality constaint for stage t, scenario j problem:
    - $E_{t-1}^j = \sum_{d(j)} \frac{p_t^{d(j)}}{p_t^j} \pi_t^{d(j)} B_t$
    - $e_{t-1}^j = \sum_{d(j)} \frac{p_t^{d(j)}}{p_t^j} \left[\pi_t \xi_t^{d(j)} + \sum_{feas} \rho_t^{d(j)} d_t^{d(j)} + \sum_{opt} \sigma_t^{d(j)} e_t^{d(j)} \right]$
    - Similar to L-shaped method, cut is $E_t^j x_t^j + \theta_t^j \geq e_t^j$ for a particular stage/scenario problem if $\theta(new) = e_{t-1}^j - E_{t-1}^j x_{t-1}^j > \theta(old)$

Example- Description

- Three stages
- Production costs \$4/unit in first stage, \$6/unit in second stage, no production opportunity in third stage
- Inventory may be carried over from one stage to another, at a cost of \$3/unit/stage
- Demand in second stage={3,5} w/ equal probability (must satisfy)
- Demand in third stage={0,3} w/ equal probability (must satisfy)
- Decision variables=x_t^j how many to produce in period t, assuming scenario j
- "Fixed Recourse" variables=I_t^j unsold goods in period t, assuming scenario j, after observing demand ξ_t^j, $t = 2, 3$

Example- LP's

- Stage 1:

$$\min 4x_1, x \geq 0$$

- $\Xi_2 = \{3, 5\}$
- Stage 2:

$$\min 3I_2^j + 6x_2^j$$
$$s.t. \quad I_2^j = x_1 - \xi_2^j$$
$$x_2^j, I_2^j \geq 0$$

- $\Xi_3 = \{0, 3\}$
- Stage 3:

$$\min 3I_3^j$$
$$s.t. \quad I_3^j = I_2^{a(j)} + x_2^{a(j)} - \xi_3$$
$$I_3^j \geq 0$$

Example- Equivalent Deterministic Problem

Model

$$\min 4x_1 + \sum_{j=1,2}\left[\frac{1}{2}\left(3l_2^j + 6x_2^j\right)\right] + \sum_{j=1,2,3,4}\left[\frac{1}{4}\left(3l_3^j\right)\right]$$

$$\begin{array}{cccccccccccc}
x_1 & l_2^1 & x_2^1 & l_2^2 & x_2^2 & l_3^1 & l_3^2 & l_3^3 & l_3^4 & & \\
-1 & 1 & & & & & & & & = & -3 \\
-1 & & & 1 & & & & & & = & -5 \\
 & -1 & -1 & & & 1 & & & & = & 0 \\
 & -1 & -1 & & & & 1 & & & = & -3 \\
 & & & -1 & -1 & & & 1 & & = & 0 \\
 & & & -1 & -1 & & & & 1 & = & -3 \\
 & all & x, l & \geq & 0 & & & & & &
\end{array}$$

Notes

- algorithm converges in finite number of steps (since finite scenarios at each stage)

Proposition 3—HW problem!!

Let scenario j' at period $t+1$ generate a feasibility constraint for its ancestor problem t/j. If that constraint is binding, then all basic feasible solutions of problem $j'/t+1$ are degenerate. Proven in Birge (1980)

- Why does this happen?

Notes

- algorithm converges in finite number of steps (since finite scenarios at each stage)

Proposition 3—HW problem!!

Let scenario j' at period $t+1$ generate a feasibility constraint for its ancestor problem t/j. If that constraint is binding, then all basic feasible solutions of problem $j'/t+1$ are degenerate. Proven in Birge (1980)

- Why does this happen?
- problem $j'/t+1$ is independent of j/t problem. How might you fix it?

Notes

- algorithm converges in finite number of steps (since finite scenarios at each stage)

Proposition 3—HW problem!!

Let scenario j' at period $t+1$ generate a feasibility constraint for its ancestor problem t/j. If that constraint is binding, then all basic feasible solutions of problem $j'/t+1$ are degenerate. Proven in Birge (1980)

- Why does this happen?
- problem $j'/t+1$ is independent of j/t problem. How might you fix it?
- Method 2 in the paper forces a basis change of x_t^j when feasibility constraint is tight.