A PRIORI OPTIMIZATION OF THE PROBABILISTIC TRAVELING SALES MAN PROBLEM

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Abstract

- Probabilistic Traveling Salesman Problem (PTSP) defined on a G=(V,E)
  - Each vertex V has a probability of being present.
  - Each edge has an associated cost or distance.
- 1st Stage : Finding a priori Hamiltonian tour on G.
  - Then a list of present vertices is revealed
- 2nd stage a priori tour is followed by skipping the absent vertices.

Abstract

- Criteria of Optimization :
- The PTSP consists of determining a 1st stage solution that minimizes the expected cost of the 2nd stage tour.
- Integer linear stochastic program.
- Solved by a branch and cut approach which:
  - relaxes some of the constraints
  - lower bounding functional on the objective function.
- Results on test cases in graphs with up to 50 vertices are reported.

Introduction

- The probabilistic traveling salesman problem (PTSP) can be defined as follows:
  - Let $G = (V, E)$ be a graph
    - $V = \{v_1, \dots , v_n\}$ is the vertex set,
    - $E = \{ (v_i, v_j) \}$ is the edge set. With each edge (v_i, v_j) is associated a nonnegative distance (cost, travel time) c_{ij}.
  - We restrict our attention to symmetrical PTSPs : all edges are undirected and defined only for $i < j$
  - we also assume that (c_{ij}) satisfies the triangle inequality $:C_{ik} + C_{kj} \geq C_{ij}$ for all i, j, k.

Assumptions

- Let $p_i > 0$ be the probability that vertex v_i is present.
  - Vertices with $p_i = 1$ are referred to as black vertices.
  - Vertices with $0 < pi < 1$ are white vertices.
- We assume there is at least one white vertex.
- The sequence of decisions and information in the PTSP :
  - In the 1st stage, a priori Hamiltonian cycle over G must be determined before any information on the present white vertices is available.
  - In the 2nd stage, information becomes available, present vertices are visited in the same order as they appear on the a priori tour, by skipping the absent vertices.
  - The PTSP then consists of determining an a priori tour to minimize the expected length of the tour actually followed in the 2nd stage.

Applications

- In delivery contexts where a set of customers has to be visited on a regular basis :
- all customers do not always require a visit.
- Since it is often impractical or undesirable to re-optimize vehicle routes from scratch every day, the delivery man will follow a master route, leaving out customers that do not require a visit.
- Such an example is provided by Bartholdi et al. (1983) in the design of a meals on wheels routing system in Atlanta. The master route of least expected cost corresponds to the optimal PTSP solution.

Model

$$\min_{x} E_{\xi} T(x,\ \xi)$$

Length of 2^{nd} stage tour

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2(v_k \in V)$$

Entering and exiting each vertex

$$\sum_{\substack{v_i, v_j \in S \\ i<j}} x_{ij} \leqslant |S| - 1(S \subset V;\ 3 \leqslant |S| \leqslant n-3)$$

Sub tour elimination

$$x_{ij} \in \{0,\ 1\} \qquad (v_i,\ v_j) \in E.$$

Notation	
x_{ij}	If edge (v_i , v_j) is used in the tour
$\xi = (\xi_i)$	is a vector of Bernouilli random variables, where (ξ_i is equal to 1 if and only if vertex v_i is present (with probability p_i).
cx	length of the a priori tour

Model-Explanations

$$\min_{x} E_{\xi} T(x, \xi)$$

Length of 2nd stage tour

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2(v_k \in V)$$

Entering and exiting each vertex

$$\sum_{\substack{v_i, v_j \in S \\ i<j}} x_{ij} \leqslant |S| - 1(S \subset V;\ 3 \leqslant |S| \leqslant n-3)$$

Sub tour elimination

$$x_{ij} \in \{0, 1\} \qquad (v_i, v_j) \in E.$$

Notation	
$R(x, \xi)$	Distance reduction in tour length in 2nd stage, as a result of skipping nodes after realization of randomness.
$Q(x, \xi)$	$-R(x, \xi)$
$Q(x)$	$E_{\xi} Q(x, \xi)$

Stochastic Form- Solution Scheme

$$\min_{x} cx + Q(x).$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2(v_k \in V)$$

$$\sum_{\substack{v_i, v_j \in S \\ i<j}} x_{ij} \leqslant |S| - 1(S \subset V;\ 3 \leqslant |S| \leqslant n-3)$$

$$x_{ij} \in \{0,\ 1\} \qquad (v_i,\ v_j) \in E.$$

- Stochastic linear program with binary decision variables.
- Could be solved using a relaxation approach and a branch and cut algorithm

Algorithm-Initiation

- Define the initial current problem as

$$\min_{x,\theta} cx + \theta$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2(v_k \in V)$$

$$0 \leq x \leq 1 \qquad L \leq \theta$$

- 3 types of relaxations:
  - Sub-tour elimination *(Will be introduced where necessary)*
  - Integrality *(Will be regained through B&B)*
  - The term Q(x) has been replaced by an approximation θ with a lower bound *L*. *(how to derive the bound would be discussed later)*
- the recourse function Q(x) is gradually approximated through the introduction of optimality cuts to be described later.

Algorithm

STEP 0. Set ν, the current iterate point, equal to 0, and $\bar{z} := \infty$. The only pendant node of the search tree corresponds to the initial current problem.

STEP 1. Select a pendant node from the list. If none exists, stop.

STEP 2. Set $\nu := \nu + 1$. Let (x^ν, θ^ν) be an optimal solution to the current problem.

STEP 3. If $cx^\nu + \theta^\nu \geq \bar{z}$ fathom the current problem and return to Step 1. Otherwise, check for any violated subtour elimination constraint (3). If one can be identified, augment the current problem accordingly, and return to Step 2.

Adding the necessary sub tour elimination constraint

Algorithm-Continued

STEP 4. Check for integrality restrictions. If one is violated, create two new branches following the usual rules; append the new nodes to the list of pendant nodes; return to Step 1.

STEP 5. Compute $Q(x^\nu)$ and $z^\nu = cx^\nu + Q(x^\nu)$. If $z^\nu < \bar{z}$, set $\bar{z} := z^\nu$.

Will be discussed later

STEP 6. If $\theta^\nu \geq Q(x^\nu)$, fathom the current node and return to Step 1. Otherwise impose one optimality cut and return to Step 2.

Will be discussed later

Optimality Cuts

Proposition 1. *The optimality cuts*

$$\theta \geqslant \tfrac{1}{2}\,(Q(x^{\nu}) - L)\left(\sum_{(v_i, v_j) \in E^{\nu}} x_{ij} - n\right) + Q(x^{\nu}) \quad (7)$$

where $E^{\nu} = \{(v_i, v_j) \in E : x_{ij}^{\nu} = 1\}$ *are valid inequalities for the PTSP.*

- 2 cases :
  - $x = x^v$ what happens ?
  - Otherwise what happens ?

Optimality Cuts

Proof. The validity of (7) follows from the fact that $\sum_{(v_i, v_j) \in E^{\nu}} x_{ij}$ takes the value n when $x = x^{\nu}$, and a value not exceeding $n - 2$ otherwise. In the first case, (7) reduces to $\theta \geqslant Q(x^{\nu})$, and in the second case, to $\theta \geqslant L$.

- These cuts guarantee that the algorithm terminates in a finite number of operations.

Example of Feasibility Cuts

- ξ can take values 1 and 2 with equal probability
- Assume x=[2 2]

$$
\begin{aligned}
-y_1 + y_2 &\le \xi - x_1, \\
y_1 + y_2 &\le 2 - x_2, \\
y_1, y_2 &\ge 0 \text{ and integer},
\end{aligned}
$$

- What should we do now?

Example of Feasibility Cuts

$$
\begin{aligned}
w &= v_1 + v_2, \\
y_1 &= 0.4 + y_2 + s_1 - v_1, \\
s_2 &= 0 - 2y_2 - s_1 + v_1 + v_2.
\end{aligned}
$$

- Where we should Branch?

Example of Feasibility Cuts

- For Y1=<0

$$
\begin{aligned}
w &= 0.4 + y_2 + s_1 + s_3 + v_2, \\
y_1 &= 0 - s_3, \\
s_2 &= 0.4 - y_2 + s_3 + v_2, \\
v_1 &= 0.4 + y_2 + s_1 + s_3.
\end{aligned}
$$

For Y1>=1

$$
\begin{aligned}
w &= 0.6 + y_2 + s_2 + s_3 + v_1, \\
y_1 &= 1 + s_3, \\
v_2 &= 0.6 + y_2 + s_2 + s_3, \\
s_1 &= 0.6 - y_2 + s_3 + v_1.
\end{aligned}
$$

- Now, How we should create Feasibility Cut?

Example of Optimality Cuts

- $\xi_1 = 8$
- X=[0 6]

$$E_{\boldsymbol{\xi}} \min\{-8y_1 - 9y_2 \text{ s. t. } 3y_1 + 2y_2 \leq \boldsymbol{\xi}, -y_1 + y_2 \leq x_1, y_2 \leq x_2, y \geq 0, \text{ integer}\}.$$

Example of Optimality Cuts

$$
\begin{aligned}
z &= -136/5 + 17s_1/5 + 11s_2/5, \\
y_1 &= 8/5 - s_1/5 + 2s_2/5, \\
y_2 &= 8/5 - s_1/5 - 3s_2/5, \\
s_3 &= 22/5 + s_1/5 + 3s_2/5,
\end{aligned}
$$

- Where we need to Branch?

Example of Optimality Cuts

- For Y1=<1

$$\begin{aligned} z &= -17 + 9s_2 + 17s_4, \\ s_1 &= 3 + 2s_2 + 5s_4, \\ y_2 &= 1 - s_2 - s_4, \\ s_3 &= 5 + s_2 + s_4, \\ y_1 &= 1 - s_4. \end{aligned}$$

For Y1>=2

$$\begin{aligned} z &= -25 + 9/2s_1 + 11/2s_4, \\ y_1 &= 2 + s_4, \\ y_2 &= 1 - s_1/2 - 3/2s_4, \\ s_3 &= 5 + s_1/2 + 3/2s_4, \\ s_2 &= 1 + s_1/2 + 5/2s_4. \end{aligned}$$

- How the Optimality Cut should be created?

COMPUTATION OF Q(x^v)

- $Q(x^\nu) = T(x^\nu) - cx^\nu$ here T(x^v) is the expected length of the tour defined by x^v
- define t(k) as the expected length from v_{i_k} to v_1 if v_{i_k} is present
- Starting with t(n + 1) = 0 and t(n) = $c_{i_n 1}$, we obtain

$$t(k) = \sum_{r=0}^{n-k} \prod_{j=1}^{r} (1 - p_{i_{k+j}}) p_{i_{k+r+1}} (c_{i_k i_{k+r+1}} + t(k + r + 1))(k = n - 1, \ldots, 1$$

$$c_{i_k i_{n+1}} = c_{i_k 1},\ p_{i_{n+1}} = p_1$$

- T(x^v) = t(1)

Computation of “L”

- Concept of Shortcut Edge
- First an Upper Bound on the expected distance reduction should be found by solving a ILP, and then L=-U

$$\textbf{UB:} \quad U = \max \sum_{i,j,k} d_{ikj}\, y_{ikj} \tag{9}$$

subject to

$$\sum_{\substack{i,j \neq k \\ i<j}} y_{ikj} = 1 \quad (k = 1, \ldots, n) \tag{10}$$

$$\sum_{k \neq i,j} y_{ikj} \leqslant 1 \quad (i, j = 1, \ldots, n;\ i < j) \tag{11}$$

$$\sum_{\substack{j<i \\ k \neq i,j}} y_{ikj} + \sum_{\substack{j>i \\ k \neq i,j}} y_{jki} \leqslant 2 \quad (i = 1, \ldots, n) \tag{12}$$

$$y_{ikj} \in \{0, 1\}$$

$$(i, j = 1, \ldots, n;\ i < j;\ k = 1, \ldots, n). \tag{13}$$

Preposition 2

- **If n>=5, U is an upper bound on R(x)**
- **Proof:**
  - Constraint 11 is valid when N>=5 !
  - Constraint 12 is hold, because the number of V generating a shortcut is at most 2
  - Triangle inequality holds, so U is a valid upper bound

Computation of L

- Using Several Relation for cons. 10, 11, 12, or "*Integrality Requirement*"
- Result: Linear Relation gives Optimal Upper bound

Computation of LB

- For any edge we define b_{ij} as LB of Value of Expected Distance:

$$b_{ij} = p_i p_j c_{ij} + \tfrac{1}{2}\, p_i(1 - p_j) \min_{k \neq j} \{c_{ik}\} + \tfrac{1}{2}\,(1 - p_i)p_j \min_{k \neq i} \{c_{kj}\}. \qquad (14)$$

- Preposition 3:
- Let $b=(b_{ij})$, then $\theta \geqslant (b - c)x$s a valid inequality for the PTSP

1. Using "Fixed Chains"

Let $H = (v_{\iota_q}, \ldots, v_{\iota_r})$ be a chain for which $x_{\iota_q \iota_{q+1}}, \ldots, x_{\iota_{r-1} \iota_r}$ have all been fixed to 1. Also, let $v_{\iota_j}, v_{\iota_k} \in \{v_{\iota_q}, \ldots, v_{\iota_r}\}$, $j \neq k$. For edge $(v_{\iota_j}, v_{\iota_k})$ to be traversed, v_{ι_j} and v_{ι_k} must be present, while $v_{\iota_{j+1}}, \ldots, v_{\iota_{k-1}}$ must be absent whenever $k > j + 1$. The expected length of $(v_{\iota_q}, \ldots, v_{\iota_r})$ is then

$$b_H = \sum_{j=q}^{r-1} \sum_{k=j+1}^{r} p_{\iota_j} p_{\iota_k} \prod_{t=j+1}^{k-1} (1 - p_{\iota_t}) c_{\iota_j \iota_k}, \tag{15}$$

where the last product is equal to 1 if $k = j + 1$, and the negative value of the savings associated with H is

$$a_H = b_H - \sum_{t=q}^{r-1} c_{\iota_t \iota_{t+1}}. \tag{16}$$

2. Free Edges Incident to Fixed Chains

Let $(v_{i_q}, \ldots, v_{i_r})$ be a fixed chain and (v_{i_r}, v_k), an edge such that v_k does not belong to any fixed chain. If v_k is present, then the expected distance traveled from a vertex of the chain into v_k, and out of v_k, is bounded below by the bracketed term of (17) computed with $p_{i_{r+1}} = p_k$, plus ½ $p_k \min_{v_t \in V \setminus F} \{c_{kt}\}$, where F is the set of vertices unreachable from v_k. If v_k is absent, then the expected distance associated with (v_{i_r}, v_k), and out of v_{i_r}, is bounded below by ½ $p_{i_r}(1 - p_k) \min_{v_t \in V \setminus T} \{c_{i_r t}\}$. It follows that

$$b_{i_r k} = p_k \left[\sum_{j=q}^{r} p_{i_j} \prod_{t=j+1}^{r} (1 - p_{i_t}) c_{i_j k} + \tfrac{1}{2} \min_{v_t \in V \setminus F} c_{kt} \right] + \tfrac{1}{2} p_{i_r} (1 - p_k) \min_{v_t \in V \setminus F} \{c_{i_r t}\}, \qquad (18)$$

2. Free Edges Incident to Fixed Chains

Now consider two fixed chains $(v_{\iota_q}, \ldots, v_{\iota_r})$ and $(v_{\iota_s}, \ldots, v_{\iota_u})$, and the free edge $(v_{\iota_r}, v_{\iota_s})$. Then, the expected distance traveled on $(v_{\iota_r}, v_{\iota_s})$ is bounded below by

$$
\begin{aligned}
b_{\iota_r \iota_s} = \sum_{j=q}^{r} \sum_{k=s}^{u} p_{\iota_j} p_{\iota_k} & \prod_{t=j+1}^{r} (1 - p_{\iota_t}) \\
& \cdot \prod_{l=s}^{k-1} (1 - p_{\iota_l}) c_{\iota_j \iota_k} \\
& + \tfrac{1}{2} p_{\iota_r} (1 - p_{\iota_s}) \min_{v_t \in V \setminus F} \{c_{\iota_r t}\} \\
& + \tfrac{1}{2} p_{\iota_s} (1 - p_{\iota_r}) \min_{v_t \in V \setminus F} \{c_{\iota_s t}\}, \qquad (19)
\end{aligned}
$$

Computational Results

- The algorithm was tested on problems ranging from 10 to 50 vertices with varying proportions of black vertices.
- ***"See Table I"*** *which has distributed.*
- In all cases vertices were randomly generated with uniform distribution.
- Results indicates that PTSPs can be solved to optimality for medium size values of n.

Computational Results

- More Random problems tend to be more difficult.
- Problems become difficult when number of Black vertices decrease, or probability of presence of white vertices becomes smaller