Stochastic Programming
Lecture 2

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Today's Class

1. One-Stage Stochastic Programs
2. Farmer Ted Example - From B & L P. 4

Outline

One-Stage Stochastic Programs

We will talk mostly about stochastic linear programs. Consider the following LP....

$$\min\{cx|Ax \geq b, x \geq 0\}$$

$x \in \Re^n$ is a vector of *decision variables*. The parameters of the problem are: $c \in \Re^n$, $b \in \Re^m$, $A \in \Re^{m \times n}$. In a stochastic linear program some or all of the parameters are random variables.

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Question: How does uncertainty in c, b, A affect the optimal solution to this problem?

One-Stage Stochastic Programs

Consider the following LP...

$$z(\xi) = \max\{\xi_1 x_1 + \xi_2 x_2 | x_1 + x_2 \le 1, 2x_1 + 0.5x_2 \le 1, x_1, x_2 \ge 0\}$$

where $\xi = (\xi_1, \xi_2)$. Let $\xi_1 \sim U(0,1)$ and $\xi_2 \sim U(0,1)$.

Question: How does uncertainty in the cost coefficients influence the optimal solution?

One-Stage Stochastic Programs

Consider the following LPs with uncertainty in the constraints...

Example 1:

$$z(\xi) = \max\{2x_1 + x_2 | x_1 + x_2 \leq \xi_1, 2x_1 + 0.5x_2 \leq \xi_2, x_1, x_2 \geq 0\}$$

where $\xi_1 \sim U(0,1)$, $\xi_2 \sim U(0,1)$

Example 2:

$$z(\xi) = \max\{2x_1 + x_2 | \xi_1 x_1 + x_2 \leq 1, \xi_2 x_1 + 0.5x_2 \leq 1, x_1, x_2 \geq 0\}$$

where $\xi_1 \sim U(0,1)$, $\xi_2 \sim U(0,1)$

Outline

Farmer Ted Example - From B & L P. 4

Farmer Ted can grow Wheat or Corn on his 500 acres (yes, I know this sounds boring):

- Farmer Ted requires 200 tons of wheat and 240 tons of corn to feed his cattle. These can be grown on his land or bought from a wholesaler
- Any production in excess of these amounts can be sold for \$170/ton (wheat) and \$150/ton (corn)
- Any shortfall must be bought from the wholesaler at a cost of \$238/ton (wheat) and \$210/ton (corn).

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Farmer Ted can also grow beets (the plot thickens!):

- Beets sell at $36/ton for the first 6000 tons
- Due to economic quotas on beet production, beets in excess of 6000 tons can only be sold at $10/ton

Farmer Ted Example - From B & L P. 4

Important information:

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting Cost ($/acre)	150	230	260
Selling Price ($/T)	170	150	36 ($< 6000T$) and 10 ($\geq 6000T$)
Purchase Price ($/T)	238	210	–
Minimum Requirement (T)	200	240	–

Total Available Land: 500 Acres

Farmer Ted Example - From B & L P. 4

Define the following decision variables....

x_1 : acres of land devoted to wheat
x_2 : acres of land devoted to corn
x_3 : acres of land devoted to beets
w_1 : tons of wheat sold
y_1 : tons of wheat purchased
w_2 : tons of corn sold
y_2 : tons of corn purchased
w_3 : tons of beets sold at favorable price
w_4 : tons of beets sold at lower price

Deterministic Formulation

The LP formulation is....

$$\min\{150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 + 210y_2 \\ - 150w_2 - 36w_3 - 10w_4\}$$

$$s.t.\ \ x_1 + x_2 + x_3 \leq 500,\ \ 2.5x_1 + y_1 - w_1 \geq 200,$$

$$3x_2 + y_2 - w_2 \geq 240,\ \ w_3 + w_4 \leq 20x_3,\ \ w_3 \leq 6000,$$

$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.$$

	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–

Total Profit: $118,600

Stochastic Programming Formulation

What if the yields are not known in advance? Assume 3 possible yield *scenarios*: (a) 20% above average (b) average and (c) 20% below average. What should the objective be now?

Stochastic Programming Formulation

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	Wheat	Corn	Sugar Beets
Surface (acres)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	–	6000
Purchase (T)	–	–	–

Table: Optimal solution for high yield (+20%) scenario. Total Profit: $167,667 (Yeah!)

	Wheat	Corn	Sugar Beets
Surface (acres)	100	25	375
Yield (T)	200	60	6000
Sales (T)	–	–	6000
Purchase (T)	–	180	–

Table: Optimal solution for high yield (-20%) scenario. Total Profit: $59,950 (Booo!)

Stochastic Programming Formulation

A model (extensive form) that minimizes expected costs is:

$$
\begin{aligned}
\min\ & 150x_1 + 230x_2 + 260x_3 + \tfrac{1}{3}(238y_1^1 - 170w_1^1 + 210y_2^1 \\
& -150w_2^1 - 36w_3^1 - 10w_4^1) + \tfrac{1}{3}(238y_1^2 - 170w_1^2 + 210y_2^2 - 150w_2^2 \\
& -36w_3^2 - 10w_4^2) + \tfrac{1}{3}(238y_1^3 - 170w_1^3 + 210y_2^3 - 150w_2^3 - 36w_3^3 - 10w_4^3) \\
s.t.\ \ & x_1 + x_2 + x_3 \le 500 \\
& 3x_1 + y_1^1 - w_1^1 \ge 200 \\
& 3.6x_2 + y_2^1 - w_2^1 \ge 240 \\
& w_3^1 + w_4^1 \le 24x_3, \quad w_3^1 \le 6000 \\
& 2.5x_1 + y_1^2 - w_1^2 \ge 200 \\
& 3x_2 + y_2^2 - w_2^2 \ge 240 \\
& w_3^2 + w_4^2 \le 20x_3, \quad w_3^2 \le 6000 \\
& 2x_1 + y_1^3 - w_1^3 \ge 200 \\
& 2.4x_2 + y_2^3 - w_2^3 \ge 240 \\
& w_3^3 + w_4^3 \le 16x_3, \quad w_3^3 \le 6000 \\
& x, y, w \ge 0.
\end{aligned}
$$

Stochastic Solution

		Wheat	Corn	Sugar Beets
First Stage	Area (acres)	170	80	250
s=1 (High)	Yield (T)	510	288	6000
	Sales (T)	310	48	6000
	Purchase (T)	–	–	–
s=2 (Average)	Yield (T)	425	240	5000
	Sales (T)	225	–	5000
	Purchase (T)	–	–	–
s=3 (Low)	Yield (T)	340	192	4000
	Sales (T)	140	–	4000
	Purchase (T)	–	48	–

Table: Optimal solution for stochastic model. Total Profit: $108,390

Stochastic Solution

This problem can be represented in a 2-stage format as follows:

$$\begin{aligned} \min\{150x_1 + 230x_2 + 260x_3 + \mathcal{Q}(x)\} \\ s.t. \quad x_1 + x_2 + x_3 \leq 500 \\ x_i \geq 0, \forall i. \end{aligned} \tag{1}$$

where $\mathcal{Q}(x) = E_\xi[Q(x,\xi)] = \sum_{s=1}^{3}(1/3)Q(x,\xi(s))$, and:

$$\begin{aligned} Q(x,\xi(s)) = \min\{238y_1^s - 170w_1^s - 150w_2^s - 36w_3^s - 10w_4^s\} \\ s.t. \quad \xi_1(s)x_1 + y_1^s - w_1^s \geq 200 \\ \xi_2(s)x_2 + y_2^s - w_2^s \geq 240 \\ -\xi_3(s)x_3 + w_3^s + w_4^s \leq 0 \\ w_3^s \leq 6000 \\ y_i^s, w_i^s \geq 0, \forall i \end{aligned}$$

Farmer Ted Example - From B & L P. 4

Interpretation of the solution to the stochastic program:

- What does the optimal solution to this stochastic program really mean?
- Notice that the optimal solution to the stochastic program is not the optimal solution for any of the 3 scenarios

Questions

- Question: How much should Farmer Ted be willing to pay for a perfect weather forecasting service?
- Question: What is the Value of the Stochastic Solution (VSS)?
- Review the other applications in Chapter 1 and Chapter 2 of B & L

Homework

Assignment 1:

- Work in groups of 2 or 3 (max) to develop a novel 2-stage stochastic linear programming model formulation
- Hand in a written description of your model (September 16)
- Present your model in class September 16
- Grading: Your grade will be weighted 50/50 based on the class evaluations and my evaluation. My evaluation will also be influenced by your written model description

Readings:

- Birge and Louveaux, Chapters 1-3
- Wets, R.J-B. 1974, “Stochastic Programs with Fixed Recourse: The Equivalent Deterministic Program”, SIAM Review, 16(3), p. 309-339.