The Vehicle Routing Problem with Stochastic Travel Times

Laporte, Louveaux, Mercure
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presented by Hamed Yarmand and Brian Piper

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Outline

Vehicle Routing Problem (VRP)

Deterministic VRP

Let $G = (V_0, E)$ be a graph where $V_0 = \{v_0, v_1, \ldots, v_n\}$ is the *vertex set* and $V = V_0 \backslash \{v_0\}$ and E is the arc set of V_0. The vertex v_0 is a depot where exactly m or at most m vehicles are based. $C = (c_{ij})$: Distance or Travel Cost Matrix. Assume C is symmetric, then E is the set of undirected edges (v_i, v_j) where $i < j$.

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$$
\begin{aligned}
\min \quad & f(m, \mathbf{x}) \\
s.t. \quad & i.\text{routes start and end at depot} \\
& ii.\text{every vertex of } V \text{ is visited exactly once by exactly one vehicle} \\
& iii.\text{some side constraints on vehicle routes}
\end{aligned}
$$

where f is a linear combination of m and $\mathbf{x}$, where $\mathbf{x} = (x_{ij})$, the route assignments decision variables.

VRP (continued)

When C satisfies the triangle inequality, $c_{ij} \leq c_{ik} + c_{kj} \ \forall i, j, k$, then constraint *ii* is not *restrictive*. (Why?)

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- Travel times
- Service times at each vertex
- Demand from each customer (Laporte, Louveaux and Mercure 1989)
- Random customers (Laporte, Louveaux and Mercure 1994)

Vehicle Routing Problem with Stochastic Travel Times (SVRP)

SVRP

Objective is to plan optimal vehicle routes with random travel time and service time.

$T = (t_{ij})$ is the travel time associated with E
$\tau = (\tau_i)$ is the stochastic service time vector defined on V
$B =$ maximum non-penalized route duration. Routes that take longer than B are penalized in proportion to the excess time.

Note that travel or service times can be continuous or discrete and independent or dependent. Dependent random variables can limit the number of scenarios and make for an easier problem to solve.

Chance Constrained Model

Objective

Minimize vehicle and routing costs while ensuring the probability of a route exceeding B is less than or equal to a given α.

Notation

$x_{ij}(i < j)$: the number of times edge (v_i, v_j) is used in the optimal solution. This variable is equal to 0 or 1 if $0 < i < j$ and to $0, 1$ or 2 if $i = 0$;

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f: the fixed cost (per planning period) of a vehicle

Chance Constrained Model (continued)

$$\min_{x,m} \; fm + \sum_{i<j} c_{ij}x_{ij}$$

$s.t.$

$$\sum_{j=1}^{n} x_{0j} = 2m \tag{1}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \quad (v_k \in V) \tag{2}$$

$$\text{Illegal route elimination constraints} \tag{3}$$

$$x_{ij} \in \{0; 1\} \quad (v_i, v_j \in V) \tag{4}$$

$$x_{0j} \in \{0; 1; 2\} \quad (v_j \in V) \tag{5}$$

$$m \geq 1 \text{ and integer} \tag{6}$$

Subtour Elimination

Standard Subtour elimination taken from the TSP.

Let $Q_S = \{(v_i, v_j) : v_i \in S, v_j \notin S \text{ or } v_i \notin S, v_j \in S, i < j\}$. Then

$$\sum_{(v_i, v_j) \in Q_S} x_{ij} \geq 2 \quad (S \subseteq V, 3 \leq |S| \leq n-3) \tag{7}$$

eliminates subtours not including v_0.

Illegal Route

Consider a route $L = (v_{i_0} = v_0, v_{i_1}, \ldots, v_{i_u}, v_{i_0})$. This is illegal if

$$P\left(\sum_{k=0}^{u}[t_{i_k i_{k+1}} + \tau_{i_k}] > B\right) > \alpha, \tag{8}$$

where $i_{u+1} = i_0$ and $\tau_{i_0} = 0$.

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where $i_{u+1} = i_0$ and $\tau_{i_0} = 0$. In this case, L can be eliminated by imposing

$$\sum_{k=0}^{u} x_{i_k i_{k+1}} \leq u. \tag{9}$$

But permutations of L may still be legal!

Assumption 1

Let $S \subset V$ and let L_S be a permutation of the elements of $S \cup \{v_0\}$, corresponding to a vehicle route.

$$\begin{aligned} t(L_S) = & \quad \text{total duration of } L_S \text{ including service time} \\ c(L_S) = & \quad \text{total cost of } L_S \\ L_S^* \in argmin\{c(L_S)\} = & \quad \text{least cost permutation of the cities of } S \cup \{v_0\} \end{aligned}$$

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\end{aligned}$$

Assumption 1

$$P(t(L_S) > s) \geq P(t(L_S^*) > s) \quad \forall s \geq B$$

Proposition 1

If L_S^* satisfies (8), and Assumption 1 is valid, then the following constraint is valid:

$$\sum_{Q_S} x_{ij} \geq 2V_\alpha(S) \quad (S \subset V) \tag{10}$$

where $V_\alpha(S)$ is a lower bound on the number of vehicles required to visit v_0 and all cities of S, so that the probability of incurring a penalty in $S \cup \{v_0\}$ does not exceed α. It can be taken as the smallest integer satisfying

$$P(t(L_S^*) > BV_\alpha(S)) \leq \alpha. \tag{11}$$

Proposition 1 (continued)

Proof.

By Assumption 1, if L_S^* is a route of least duration, there is no other permutation L_S such that (11) would be satisfied with a strictly smaller value of $V_\alpha(S)$. So at least $V_\alpha(S)$ vehicles are required to visit v_0 and all cities of S, and twice as many edges must link S and its complement.

The second equation computes the probability that at least one vehicle will incur a penalty while visiting $S \cup \{v_0\}$. Compares the minimum time required by all vehicles to visit $S \cup \{v_0\}$, to the total time $V_\alpha(S)$ vehicles can travel in $S \cup \{v_0\}$ without incurring a penalty. $\square$

Three-Index Recourse Model

$$
\begin{array}{rl}
m: & \text{an upper bound on the number of vehicles} \\
x_{ijl}(i<j): & \text{the number of times (0, 1 or 2) edge } (v_i, v_j) \text{ is} \\
& \text{traversed by vehicle } l \text{ in the optimal solution} \\
x = & (x_{ijl}) \\
z_{kl} = & \begin{cases} 1 & \text{if } v_k \text{ is visited by vehicle } l \\ 0 & \text{otherwise} \end{cases} \\
\xi: & \text{a vector of random variables for travel and service} \\
& \text{times, specific realizations given with } \omega \\
\Xi: & \text{the support of } \xi\text{, assumed finite in this model} \\
y_l(\omega): & \text{excess duration of route } l \text{ in scenario } \xi
\end{array}
$$

Three-Index Recourse Model (continued)

c_{ijl} : the travel cost of vehicle l on edge (v_i, v_j)

$t_{ijl}(\omega)$: the travel time of vehicle l on edge (v_i, v_j) in scenario ω

$\tau_{il}(\omega)$: the service time of vertex v_i by vehicle l in scenario ω

β_l : the (positive) unit penalty cost for excess duration on the route traveled by vehicle l

f_l : the fixed cost of vehicle l

B_l : the maximum time for route l, over which a penalty is incurred

Three-Index Recourse Model Formulation

$$\min_{x,z} \sum_{l=1}^{m} f_l z_{0l} + \sum_{l=1}^{m} \sum i < j c_{ijl} x_{ijl} + E_\xi \left(\sum_{l=1}^{m} \beta_l y_l(\omega) \right)$$

$$s.t. \sum_{l=1}^{m} z_{kl} = 1 \quad (v_k \in V)$$

$$\sum_{j=1}^{n} x_{0jl} = 2z_{0l} \quad (l = 1, \dots, m)$$

$$\sum_{i<k} x_{ikl} + \sum_{j>k} x_{kjl} = 2z_{kl} \quad (v_k \in V; l = 1, \dots, m)$$

$$\sum_{i,j \in S, i<j} x_{ijl} \leq |S| - 1$$

$$(S \subset V, 3 \leq |S| \leq n-3; l = 1, \dots, m)$$

Three-Index Recourse Model Formulation (continued)

$$B_l - \sum_{i<j} t_{ijl}(\omega) x_{ijl} - \frac{1}{2} \sum_{i<j} (\tau_{il}(\omega) - \tau_{jl}(\omega)) x_{ijl} + y_l(\omega) \geq 0$$

$$(l = 1, \ldots, m, \forall \omega)$$

$$x_{ijl} \in \{0, 1\} \quad (v_i, v_j \in V; l = 1, \ldots, m)$$

$$x_{0jl} \in \{0, 1, 2\} \quad (v_j \in V; l = 1, \ldots, m)$$

$$z_{kl} \in \{0, 1\} \quad (v_k \in V_0; l = 1, \ldots, m)$$

$$y_l(\omega) \geq 0 \quad (l = 1, \ldots, m; \forall \omega)$$

Two-Index Recourse Model (Notation Changes)

Decision Variable

$x_{ij}(i < j)$:	the number of times (0, 1, or 2) edge (v_i, v_j) is traversed
$x =$	(x_{ij})
m	can be a constant or decision variable indicating maximum number of vehicles available

ξ: vector of random variables corresponding to travel and service times

Ξ: support of ξ. Not necessarily assumed to be finite

y: the total expected excess duration of a first stage solution x

β: the (positive) unit cost of excess duration

Two-Index Recourse Model

$$\min_{x,m,y} fm + \sum_{i<j} c_{ij}x_{ij} + \beta y$$

$$s.t. \sum_{j=1}^{n} x_{0j} = 2m \tag{12}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \quad (v_k \in V) \tag{13}$$

$$\sum_{(v_i,v_j)\in Q_S} x_{ij} \geq 2 \quad (S \subseteq V, 3 \leq |S| \leq n-3) \tag{14}$$

$$x_{ij} \in \{0;1\} \quad (v_i, v_j \in V) \tag{15}$$

$$x_{0j} \in \{0;1;2\} \quad (v_j \in V) \tag{16}$$

$$m \geq 1 \text{ and integer} \tag{17}$$

Calculation of Excess Duration

No linear continuous expression of y in terms of x exists. But, the SVRP has a finite number of first stage solutions, index these by r.

$S_r =$ $\{(v_i, v_j) : v_i, v_j \in V$ and (v_i, v_j) belongs to the r^{th} feasible SVRP solution$\}$

θ_r : a constant equal to the total expected excess duration of solution r.

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Then the following constraints can be added:

$$y \geq \theta_r \left[\sum_{(v_i, v_j) \in S_r} x_{ij} - (n - m - 1) \right] \quad \forall r \tag{18}$$

$$y \geq 0 \tag{19}$$

Why is constraint (18) valid?

Constraint Types I

i. *degree constraints* which specify the vehicles used at various vertices and the degree of each vertex.

$$\sum_{j=1}^{n} x_{0j} = 2m$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \quad (v_k \in V)$$

ii. *lower and upper bounds* on the variables

$$y \geq 0$$

iii. *integrality requirements* on the variables

Constraint Types II

iv. *illegal route and subtour elimination*

$$\sum_{(v_i,v_j)\in Q_S} x_{ij} \geq 2 \quad (S \subseteq V, 3 \leq |S| \leq n-3)$$

$$\sum_{k=0}^{u} x_{i_k i_{k+1}} \leq u.$$

v. *lower bound on penalties* for excess route durations

$$y \geq \theta_r \left[\sum_{(v_i,v_j)\in S_r} x_{ij} - (n-m-1) \right] \quad \forall r$$

Algorithm I

Step 0 Let z^* be the cost of the best known feasible solution. If no solution is known, set $z^* := \infty$. Define a first current problem as the relaxed problem containing constraints (i) and (ii). Insert the current problem in a list.

Step 1 If the list is empty, print the best known solution and stop. Otherwise, select a problem from the list.

Step 2 Solve the current problem and let z be the value of its optimal solution. If $z \geq z^*$, fathom the current problem and go to Step 1.

Step 3 If the solution is not integer, create subproblems by branching on a fractional variable, insert them in the list and go to Step 1.

Algorithm II

Step 4 At an integer solution, check first for subtours, and then for illegal routes. If any violation is detected, add the appropriate illegal route of subtour elimination constraints to the current problem and go to Step 2.

Step 5 At a first stage feasible solution, let $z^{'}$ be the value of z, plus the value of expected penalty for excess duration. If $z \geq z^{*}$, fathom the current problem and go to Step 1. If $z^{'} < z^{*}$, update the best known solution and set $z^{*} := z^{'}$. Introduce in the current problem the appropriate type (v) constraint and go to Step 2.

Algorithm Comments

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Specific Modifications for each model

Chance Constraints (v) and so step 5, does not apply.

Three-Index Whenever a constraint (v) is generated for a given vehicle, it is generated for all vehicles.

Two-Index A more accurate lower bound on y can be obtained in Step 0 (see Proposition 3).

Proposition 2:

Assume C satisfies the triangle inequality and consider an optimal solution to the current problem containing no subtours identified in Step 4 of the algorithm. Let L_S be a route corresponding to this optimal solution. Then this solution is the least cost way of visiting all vertices of S.

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Proof.

The current solution is feasible and optimal for the m-TSP at the current node of the search tree. Since C satisfies the triangle inequality, each vehicle route L_S taken separately is also optimal: (i) permuting or partitioning it cannot decrease the total distance required to serve $S \cup v_0$; (ii) moreover, sharing its vertices between several routs including vertices in $V \backslash S$ cannot yield a shorter distance since for any vertex sequence $(v_{i_1}, v_{i_2}, v_{i_3})$ on a route L with $v_{i_2} \notin S$, a route of equal or shorter distance can be obtained by deleting v_{i_2} and connecting v_{i_1} to v_{i_3}. Hence, there is no less costly way of visiting all vertices of $S \cup v_0$. □

Proposition 3

A valid lower bound on y is given by

$$y \geq E_\xi \left(\sum_{i<j} t_{ij}^\xi x_{ij} + \frac{1}{2} \sum_{i<j} (\tau_i^\xi + \tau_j^\xi) x_{ij} - mB \right)^+ \tag{20}$$

where $(x)^+ = max(x, 0)$.

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where $(x)^+ = max(x, 0)$. This is equivalent to

$$y \geq \sum_{k=1}^{|\Xi|} p^k y^k \tag{21}$$

$$y^k \geq \sum_{i<j} t_{ij}^k x_{ij} + \frac{1}{2} \sum_{i<j} (\tau_i^k + \tau_j^k) x_{ij} - mB \quad (k = 1, \ldots, |\Xi|) \tag{22}$$

$$y^k \geq 0 \quad (k = 1, \ldots, |\Xi|) \tag{23}$$

Example

i	v_i	v_0	v_1	v_2	v_3	τ_i
0	v_0	-	3	5	4	0
1	v_1	3	-	6	2	2
2	v_2	5	6	-	7	3
3	v_3	4	2	7	-	1

Table: Service and Travel Times/Costs