IO543: Scheduling

Lecture 19

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Today

- Reminder: Project summaries are due today.
- More on single machine scheduling problems:
  - $1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$ for two customers
  - Value of the stochastic solution (VSS)

Appointment Scheduling

Appointment Scheduling Problem: Given a specific sequence of customers at what time should each customer arrive at a stochastic server?

The goal is to minimize the weighted sum of expected customer waiting and server idling:

$$1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$$

2 Customer Appointment Scheduling Problem:

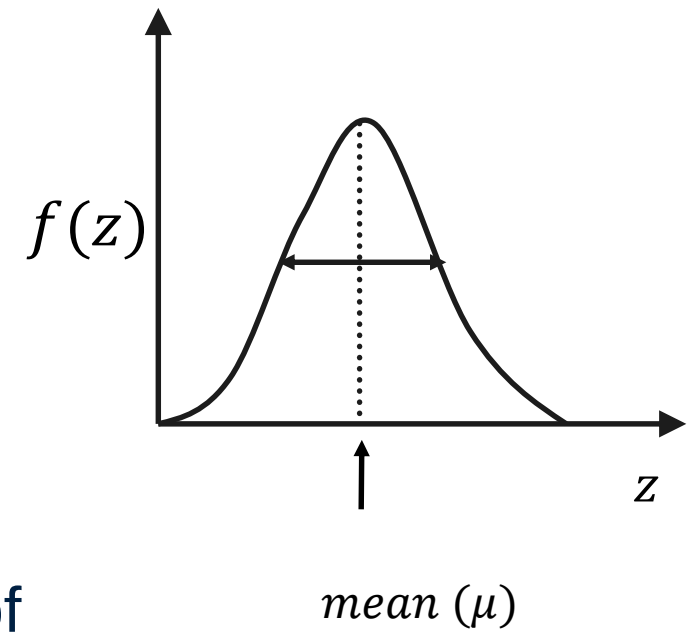
Model Parameters:

c^w = waiting cost per unit time

c^s = idling cost per unit time

Z = random service time

x = decision variable for the time allotted to customer 1 (arrival time of customer 2)



Stochastic Optimization Formulation:

$$\min_x \{E_\omega[c^w \max(0, Z - x) + c^s \max(0, x - Z)]\}$$

Customer 2 Waiting

Idling between Customer 1 And Customer 2

Uncertainty makes things difficult

Much easier to solve the **mean value problem**:

$$\min\{[c^w \max(0, E_\omega[Z] - x) + c^s \max(0, x - E_\omega[Z])]\}$$

Are these formulations the same?

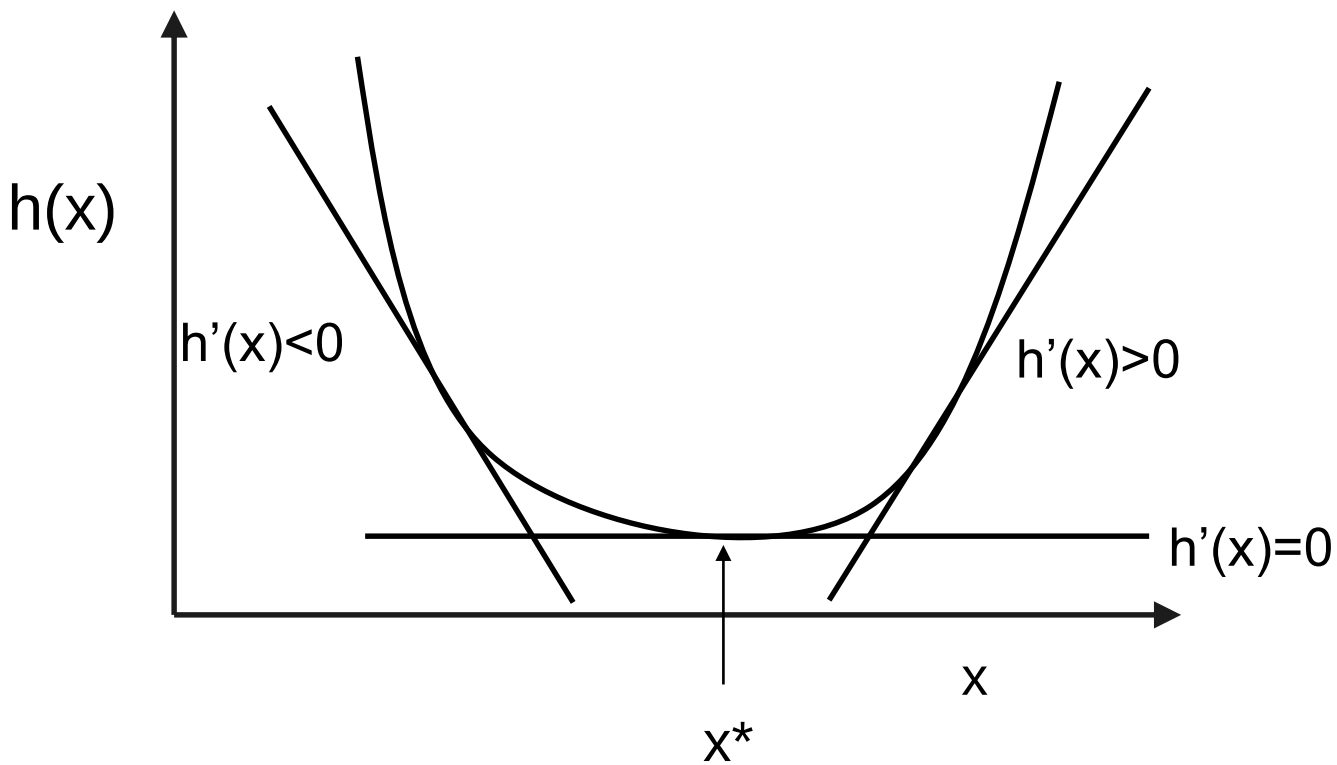
Example: Michigan Theatre

<u>Problem:</u> The Michigan Theatre schedules its first evening movie at 7pm. The total time for the showing, clearing theater, and setup for the next movie is exponentially distributed with a mean of 100 minutes. The manager estimates a loss of \$2/minute for the theater sitting idle between movies, and a loss of \$5/minute that the second movie starts late.

What time should the second movie be scheduled?

$$x^* = \arg\min\{E_\omega[5\max\{0, Z - x\} + 2\max\{0, x - Z\}]\}$$

Satisfy "first-order optimality conditions" for a **convex minimization problem**



$1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$

The optimal solution occurs when the 1st derivative of the objective function is zero:

$$h(x) = \min\{E_Z[c^w \max(0, Z - x) + c^s \max(0, x - Z)]\}$$

$$\boxed{\frac{dh(x)}{dx} = -c^w \Pr(Z - x > 0) + c^s \Pr(x - Z > 0)}$$

$$Optimality\ Condition: \ \frac{dh(x)}{dx} = 0$$

$1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$

Setting the first derivative to zero:

$$c^w \Pr(Z - x^* > 0) - c^s \Pr(x^* - Z > 0) = 0$$

and letting $F(x) = \Pr(Z \leq x)$ denote the cumulative distribution function:

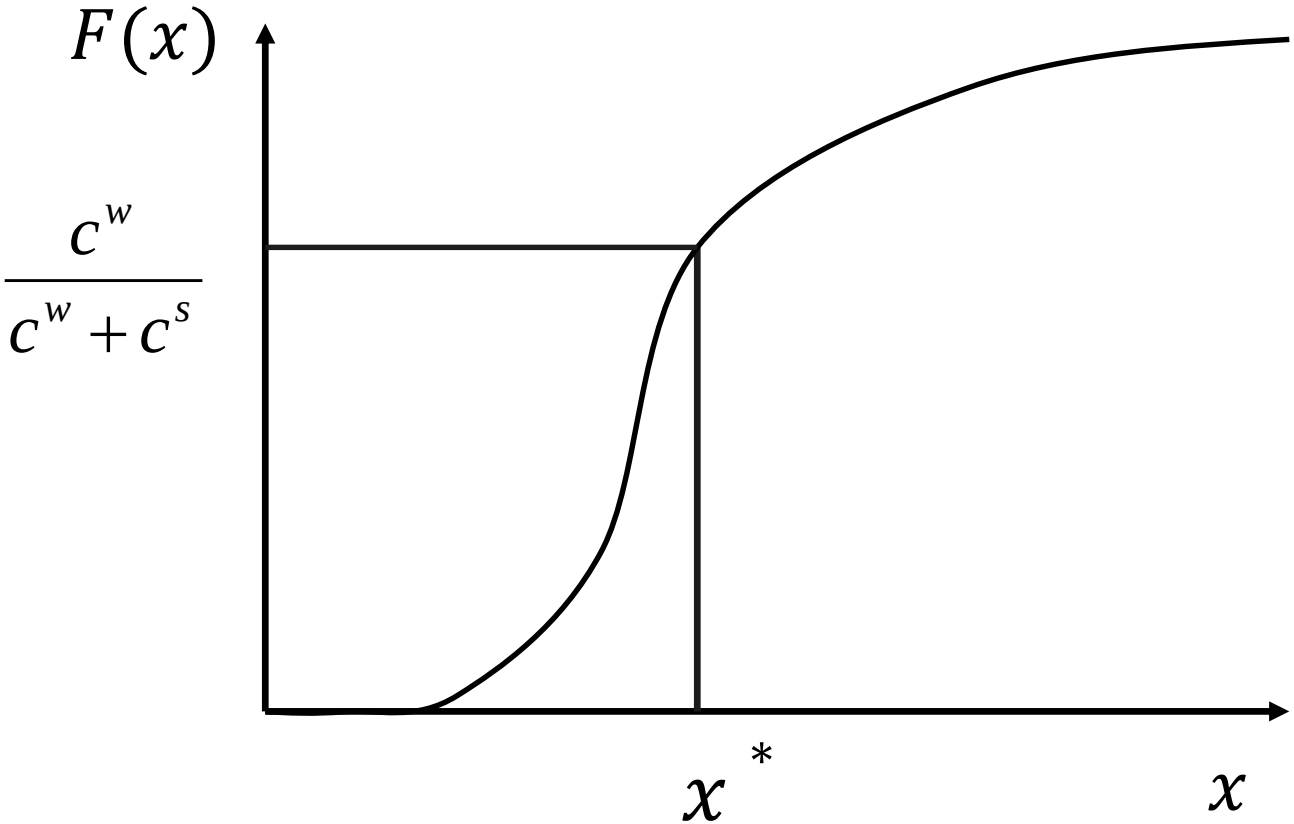
$$c^w (1 - F(x^*)) - c^s F(x^*) = 0$$

therefore:

$$F(x^*) = \frac{c^w}{c^w + c^s} \longrightarrow x^* = F^{-1}\left(\frac{c^w}{c^w + c^s}\right)$$

Michigan Theatre

Graphical illustration of solution to $1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$:



Example: Michigan Theatre

The probability density function for movie 1 time is

$$f(z) = \lambda e^{-\lambda z} = \left(\frac{1}{100}\right) e^{-(1/100)z}$$

$$F(a) = P(Z \le a) = \int_0^a \left(\frac{1}{100}\right) e^{-(1/100)z} dz$$

$$= \left[-e^{-(1/100)z}\right]_0^a = 1 - e^{-(1/100)a}$$

From the problem data it follows that:

$$F(x^*) = \frac{c^w}{c^w + c^s} = \frac{5}{5+2} = 0.714$$

Example: Michigan Theatre

Given $F(x^*) = 0.714$, solve for x^*:

$$1 - e^{-(1/100)x^*} = 0.714$$

$$1 - 0.714 = e^{-(1/100)x^*}$$

$$\ln(0.286) = -\left(\frac{1}{100}\right)x^*$$

$$x^* = 125.2$$

Therefore the theatre should plan to start the second movie at 9:05 pm. <u>Well above the average of 100 minutes!</u>

Michigan Theatre

EVPI is the difference between the cost of x^* and the **clairvoyant (wait and see) solution**

If the theatre manager knows in advance how long movie 1 will take then the optimal time is $x(\omega) = z(\omega)$, for all ω

$$EVPI = E_\omega[5\max\{0, Z(\omega) - x^*\} + 2\max\{0, x^* - Z(\omega)\}]$$
$$-E_\omega[5\max\{0, Z(\omega) - x(\omega)\} + 2\max\{0, x(\omega) - Z(\omega)\}]$$

$$= 250.6$$

Value of the Stochastic Solution

- The **Value of the Stochastic Solution (VSS)** is a measure of the importance of solving a stochastic scheduling problem

- VSS compares the optimal schedule for the mean value problem to the optimal schedule for the stochastic problem

$$VSS = E_{\omega}[h(s^{mv}, \omega)] - E_{\omega}[h(s^{*}, \omega)]$$

Example: VSS

The Michigan Theatre **mean value problem** is

$$\min\{5\max(0, E_\omega[Z] - x) + 2\max(0, x - E_\omega[Z])\}$$

it follows that $x^{mv} = E_\omega[Z] = 100$. Therefore

$$VSS = E_\omega[5\max\{0, Z - x^{mv}\} + 2\max\{0, x^{mv} - Z\}]$$

$$- \quad E_\omega[5\max\{0, Z - x^*\} + 2\max\{0, x^* - Z\}]$$

$$= 6.91$$

See ctools for complete solution.

Guest Lecture next Class

Next class we will have a guest lecture on appointment scheduling with the number of customers is more than 2

This will introduce the important topic of **stochastic programming** a general approach to optimization which can be applied to many kinds of scheduling problems

Good Background Reading: “An introductory tutorial on stochastic programming”, by Suvrajeet Sen and Julia Higle (available under background reading on ctools)