Proof of Johnson's Rule:

By contradiction: Suppose another schedule is optimal. Then there is a pair of adjacent jobs, j and k (j→k), that satisfy one of the following:

(i) j belongs to Set II and job k to Set I
(ii) j and k in Set I and $p_{1j} > p_{1k}$
(iii) j and k in Set II and $p_{2j} < p_{2k}$

Now show that C_{max} is lower in all 3 cases if j and k are interchanged. Assume that there are jobs l and k such that

$$l \rightarrow j \Rightarrow k \rightarrow m$$

Let C_{ij}, be completion time for job j on machine i and let C'_{ij} be completion time after interchange of j and k:

$$l \rightarrow k \Rightarrow j \rightarrow m$$

The interchange does not affect completion time of job m on machine 1, so just need to focus on machine 2. Therefore show $C'_{2j} \leq C'_{2k}$ under any of cases (i) – (iii);

Original schedule ($l \rightarrow j \rightarrow k \rightarrow m$):

$C_{2k} = \max(\max(C_{2l}, C_{1l} + p_{1j}) + p_{2j}, C_{1l} + p_{1j} + p_{1k}) + p_{2k}$

$$= \max(\overset{\mathcal{A}}{C_{2l} + p_{2k} + p_{2j}}, \overset{\mathcal{B}}{C_{1l} + p_{1j} + p_{2j} + p_{2k}}, \overset{\mathcal{C}}{C_{1l} + p_{1j} + p_{1k} + p_{2k}})$$

After interchange ($l \rightarrow k \rightarrow j \rightarrow m$):

$$C'_{2j} = \max(\overset{\mathcal{D}}{C_{2l} + p_{2k} + p_{2j}}, \overset{\mathcal{E}}{C_{1l} + p_{1k} + p_{2k} + p_{2j}}, \overset{\mathcal{F}}{C_{1l} + p_{1k} + p_{1j} + p_{2j}})$$

Under condition (i) $p_{1j} > p_{2j}$ and $p_{1k} < p_{2k}$. Comparing terms we have $\mathcal{A} = \mathcal{D}$, $\mathcal{F} \leq \mathcal{B}$ ($p_{1k} < p_{2k}$), $\mathcal{E} < \mathcal{C}$ ($p_{2j} < p_{1j}$)

$\Longrightarrow$ Under condition (i) $C'_{2j} \leq C_{2k}$ (therefore contradiction)

A similar approach can be used to prove contradiction under cases (ii) and (iii)