IOE543: Scheduling

Lecture 20

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My Background

- PhD candidate, IEM, Shanghai Jiao Tong Univ., China, since 2012
- Visiting student, IOE, since 2014
- Research interests: stochastic programming, applications in appointment scheduling and surgery scheduling

Today

- More on single machine scheduling problems:
  - $1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$ for multiple (n>2) customers
  - Stochastic programming model

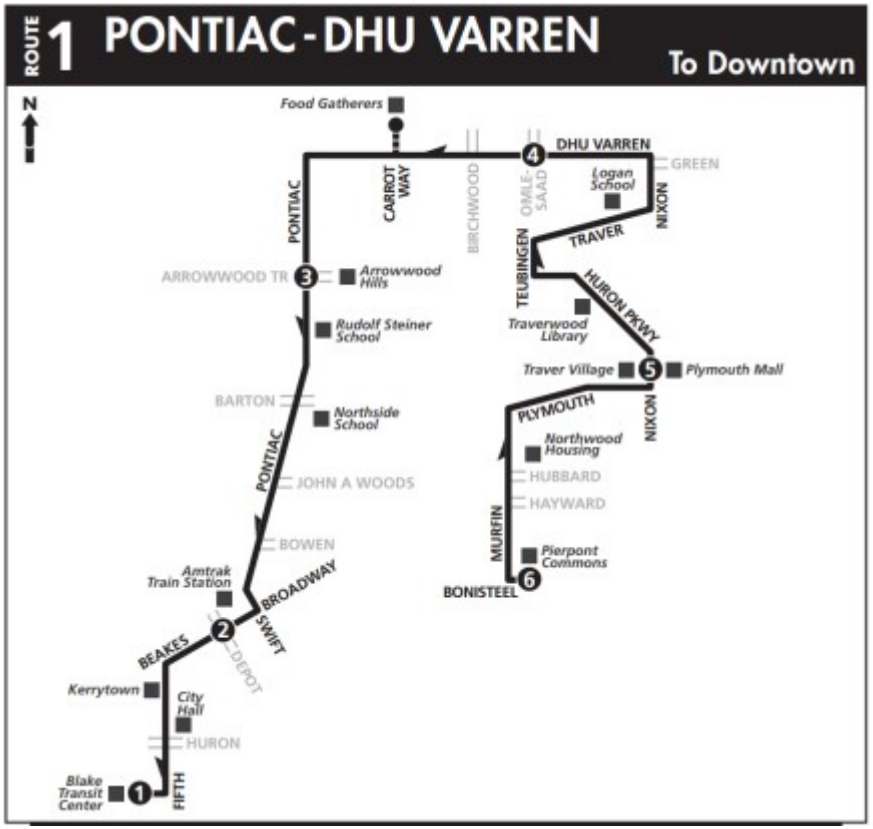
Example: bus schedule

Consider the following instance of a bus scheduling problem as $1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$:

The bus moves at a random speed

Passengers need to know the schedule in advance

Classify this as a scheduling problem



Stochastic Programs

Stochastic linear programs are linear programs with random model parameters

Linear Program:

$$\min\{cx \mid Ax = b, x \geq 0\}$$

In the **stochastic counterpart** $c, A,$ and b could be random variables

Questions we might want to answer:

- What is the optimal solution that minimizes expected cost?
- What is the EVPI? VSS?

Stochastic Programs

Two-stage stochastic linear programs involve two types of decisions:

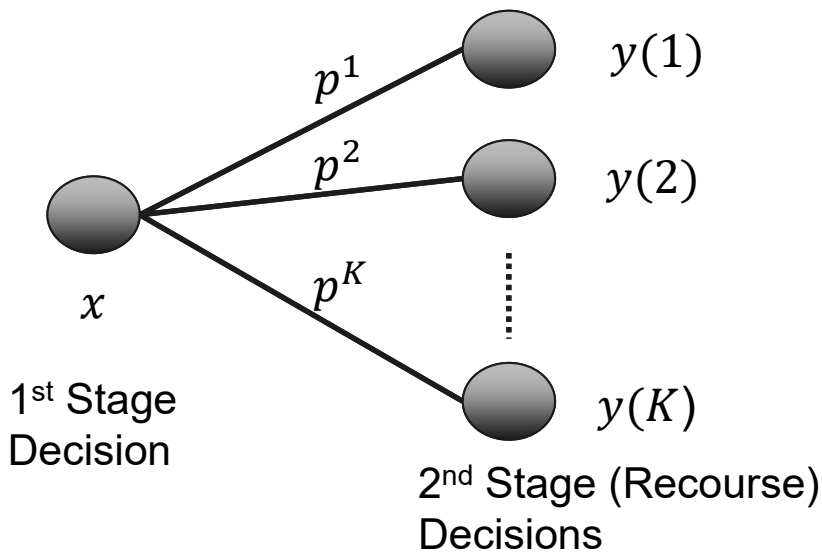
- First stage decisions: x
- Second stage decisions: $y(\omega)$

First stage decisions are made **<u>before</u>** the random variables are observed; second stage (recourse) decisions are made **<u>after</u>** the random variables are observed

$$x \rightarrow \omega \rightarrow y(\omega)$$

Two-Stage Stochastic Program

Decision Process



There are K possible scenarios

First Stage Problem: $$\min\{cx + \sum_{\omega=1}^{K} p(\omega)Q(x,\omega) \mid Ax = b, x \geq 0\}$$

Second Stage Problem: $$\mathrm{Q}(\mathrm{x},\omega) = \min\{q(\omega)y(\omega) \mid T(\omega)x + W(\omega)y(\omega) = h(\omega), y(\omega) \geq 0\}$$

Example: $1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$

$1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$ can be written as the following **nonlinear** stochastic optimization problem:

$$\min\{E_\omega[c^w \max(0, Z - x) + c^s \max(0, x - Z)]\}$$

Or alternatively as the following two-stage stochastic **linear** program:

$$\begin{aligned}
&\min \textstyle\sum_\omega^K p(\omega)(c^w w(\omega) + c^s s(\omega)) \\
&\quad s.t. \\
&\qquad w(\omega) - s(\omega) = z(\omega) - x, \qquad for\ all\ \omega \\
&\qquad w(\omega), s(\omega) \geq 0, for\ all\ \omega
\end{aligned}$$

Example: $1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$

Assume that there are two jobs and the cost of waiting is $c^w = 1$ and the cost of machine idling is $c^s = 2$.

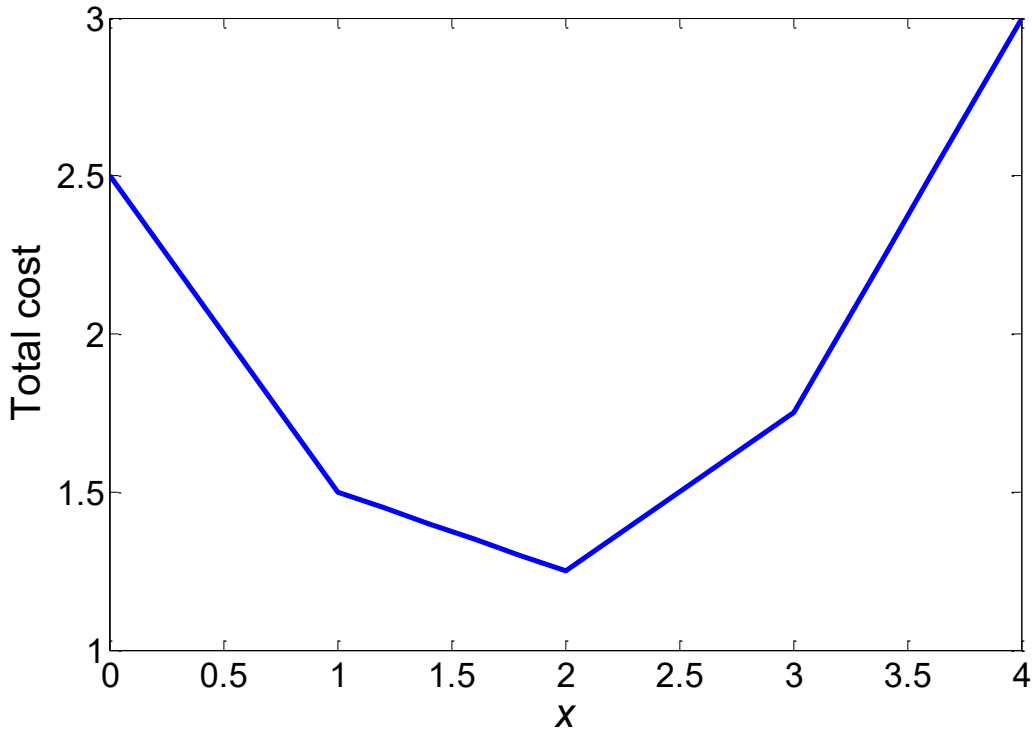
The processing time is a random variable, Z, that has a discrete uniform distribution U(1,4)

Scenario (ω)	**1**	**2**	**3**	**4**
p(ω)	¼	¼	¼	¼
Z(ω)	1	2	3	4

Formulate this problem as a stochastic program (in class)

Example: $1 \mid appt \mid (c^w \sum w_j + c^s \sum s_j)$

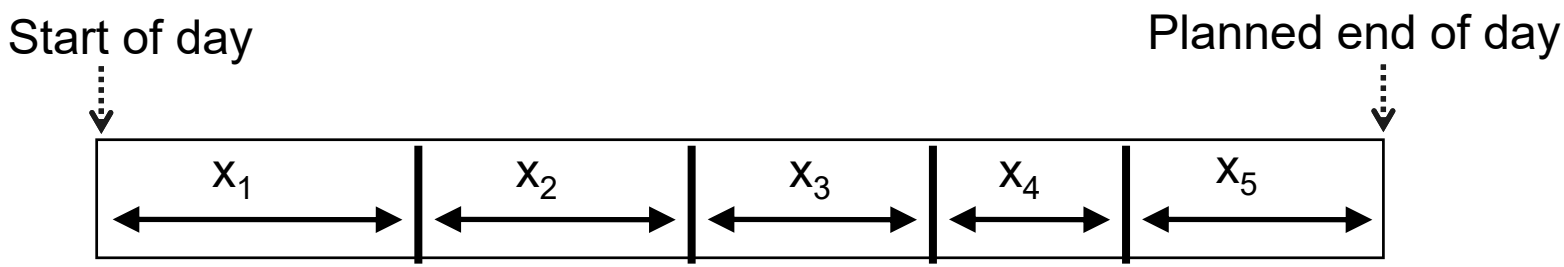
The cost function is piecewise linear when Z has a discrete uniform distribution.



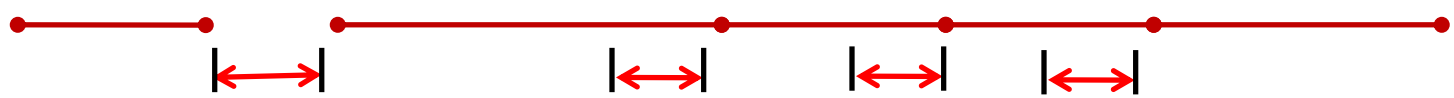
$1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)\ for\ n > 2$

Set appointment times, a_j, to minimize the total expected cost of waiting and idling: $(c^w \sum w_j + c^s \sum s_j)$

Appointment times defined by **job allowances**: $a_j = \sum_{k=1}^{j-1} x_k$



Sample scenario:



$1\,|appt\,|(c^w\sum w_j + c^s\sum s_j)$

$1\,|appt\,|(c^w\sum w_j + c^s\sum s_j)$ can be expressed as the following nonlinear stochastic optimization problem:

$$\min_x \{ \underbrace{\sum_{j=1}^{n} c^w E_\omega[w_j]}_{\text{Cost of Waiting}} + \underbrace{\sum_{j=1}^{n} c^s E_\omega[s_j]}_{\text{Cost of Idling}} \}$$

$$w_j(\omega) = \max(w_{j-1}(\omega) + Z_{j-1}(\omega) - x_{j-1}, 0),\ j = 2,..,n-1$$

$$s_j(\omega) = \max(-w_{j-1}(\omega) - Z_{j-1}(\omega) + x_{j-1}, 0),\ j = 2,...,n-1$$

Stochastic Program

$1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$ can be expressed as the following two-stage stochastic linear program:

$$\min_x \mathrm{E}_\omega [\sum_{j=2}^{n} c_j^w w_j(\omega) + \sum_{j=2}^{n} c_j^s s_j(\omega)]$$

$$s.t.$$

$$w_2(\omega) - s_2(\omega) = z_1(\omega) - x_1, for\ all\ \omega$$

$$-w_2(\omega) + w_3(\omega) - s_3(\omega) = z_2(\omega) - x_2,\ \ for\ all\ \omega$$

$$\vdots$$

$$-w_{n-1}(\omega) + w_n(\omega) - s_n(\omega) = z_{n-1}(\omega) - x_{n-1}, for\ all\ \omega$$

$$w_j(\omega), s_j(\omega) \geq 0,\ \ for\ all\ \omega, j$$

"Curse of Dimensionality"

As the number of random variables in a stochastic scheduling problem increases **the problem can become unsolvable**.

Example: Consider an instance of $1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$ with 9 customers and independent and identically distributed processing times with **discrete uniform distribution**: $U(1, 10)$

How many **scenarios** are there?

How many **decision variables** in the stochastic program?
How many **constraints**?

Monte Carlo Simulation

- **Monte Carlo** simulation involves random sampling to obtain numerical estimates

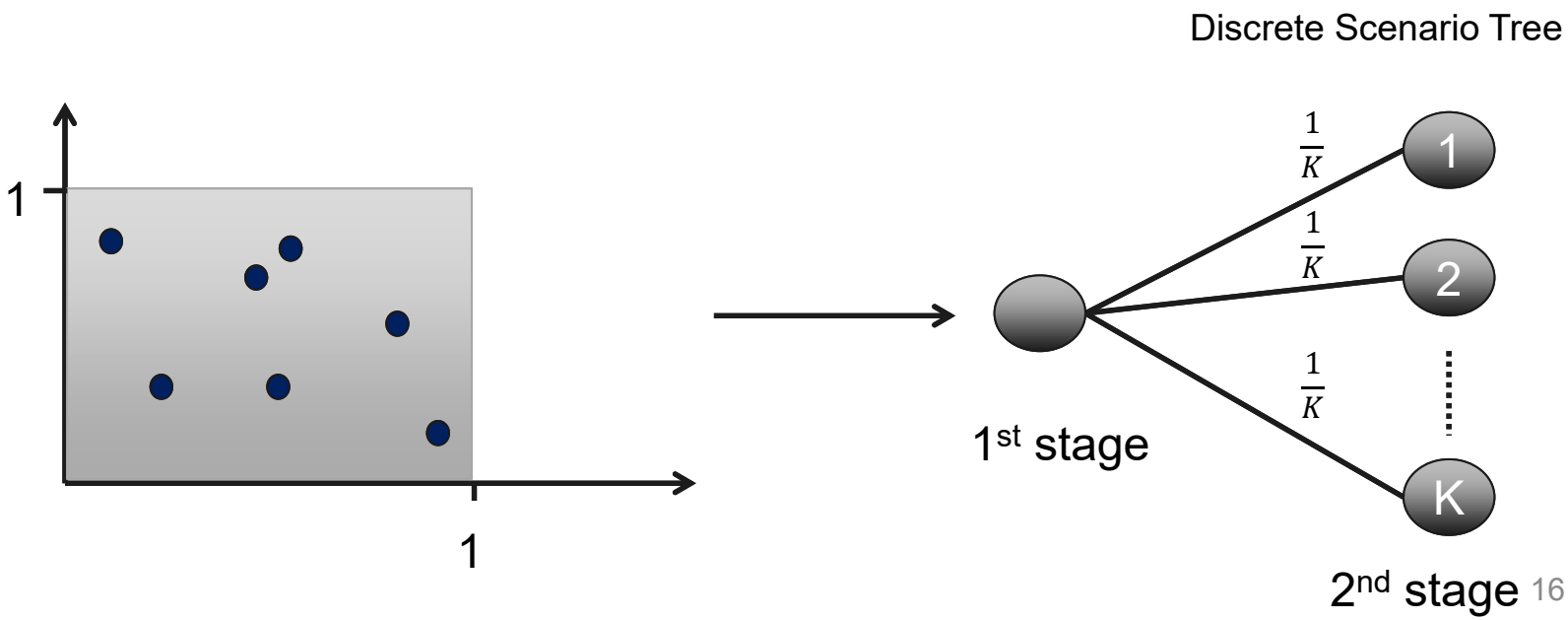
- MC Simulation has many applications, including stochastic programming where it is used to estimate probability distributions

Stanislaw Ulam

Monte-Carlo Sampling

In most cases stochastic programs with continuous random variables (e.g. Normal, Uniform, Exponential) cannot be solved exactly. The most common approximation method is **Monte-Carlo Sampling**

Example: Consider two processing times: $Z_j \sim U(0,1), \quad j = 1,2$



Stochastic Program

$1\,|appt\,|(c^w \sum w_j + c^s \sum s_j)$ can be expressed as the following **sample average approximation (SAA):**

$$\min \sum_{\omega=1}^{K} \frac{1}{K} \left(\sum_{j=2}^{n} c_j^w w_j(\omega) + \sum_{j=2}^{n} c_j^s s_j(\omega)\right)$$

$s.t.$

$$w_2(\omega) - s_2(\omega) = z_1(\omega) - x_1, for\ \omega = 1,..,K$$

$$-w_2(\omega) + w_3(\omega) - s_3(\omega) = z_2(\omega) - x_2,\ \ for\ \omega = 1,..,K$$

$$\vdots$$

$$-w_{n-1}(\omega) + w_n(\omega) - s_n(\omega) = z_{n-1}(\omega) - x_{n-1}, for\ \omega = 1,..,K$$

$$w_j(\omega), s_j(\omega) \geq 0,\ \ for\ all\ \omega, j$$

Statistical Bounds

When **statistical sampling** is used to estimate the solution to a stochastic program, **confidence intervals** must be estimated

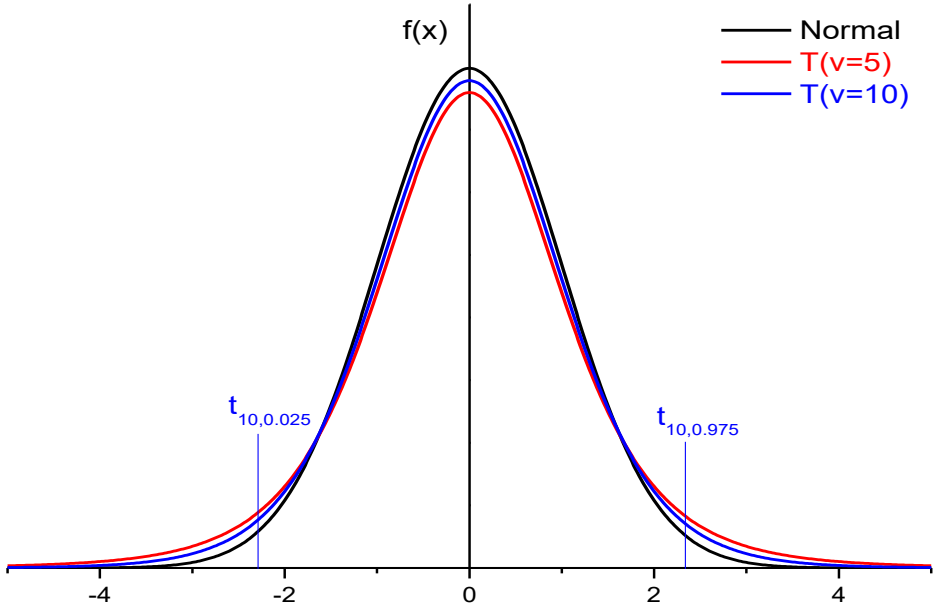
Confidence Intervals:

Step 1: Construct N independent SAAs each with K scenarios and solve each to get the optimal solution with objective $Q_n^*,\ n = 1, \dots, N$.

Step 2: Approximate $(1-\alpha)100\%$ confidence interval as:

$$[\overline{Q^*} - t_{N-1,\frac{\alpha}{2}} \frac{\sqrt{Var(Q^*)}}{\sqrt{N}}\ ,\quad \overline{Q^*} + t_{N-1,\frac{\alpha}{2}} \frac{\sqrt{Var(Q^*)}}{\sqrt{N}}]$$

Student's t distribution



Estimate the normal when the sample size is small and population standard deviation is unknown.

T-distributions are well bell-shaped and symmetric, but have 'fatter' tails than the normal.

Example

Assume that there are seven jobs and the cost of waiting is c_j^w= 1 and the cost of machine idling is c_j^s = 1.

The processing times are random variables

$$Z_j(\omega) \sim U(0,1)$$

Construct $N = 10$ independent SAAs each with $K = 1{,}000$ scenarios

Expected costs

n	$Q_n^*(x^*)$	$Q_n(\hat{x})$
1	1.661	2.163
2	1.651	2.164
3	1.625	2.193
4	1.672	2.265
5	1.653	2.196
6	1.636	2.213
7	1.634	2.138
8	1.674	2.235
9	1.638	2.218
10	1.667	2.252

$\boldsymbol{x}^*$: *optimal solution,*

$\hat{\boldsymbol{x}}$: *mean-value solution*

95% confidence interval:

$$Q^* \in [\overline{Q^*} - t_{N-1,\frac{\alpha}{2}} \frac{\sqrt{Var(Q^*)}}{\sqrt{N}}, \overline{Q^*} + t_{N-1,\frac{\alpha}{2}} \frac{\sqrt{Var(Q^*)}}{\sqrt{N}}]$$

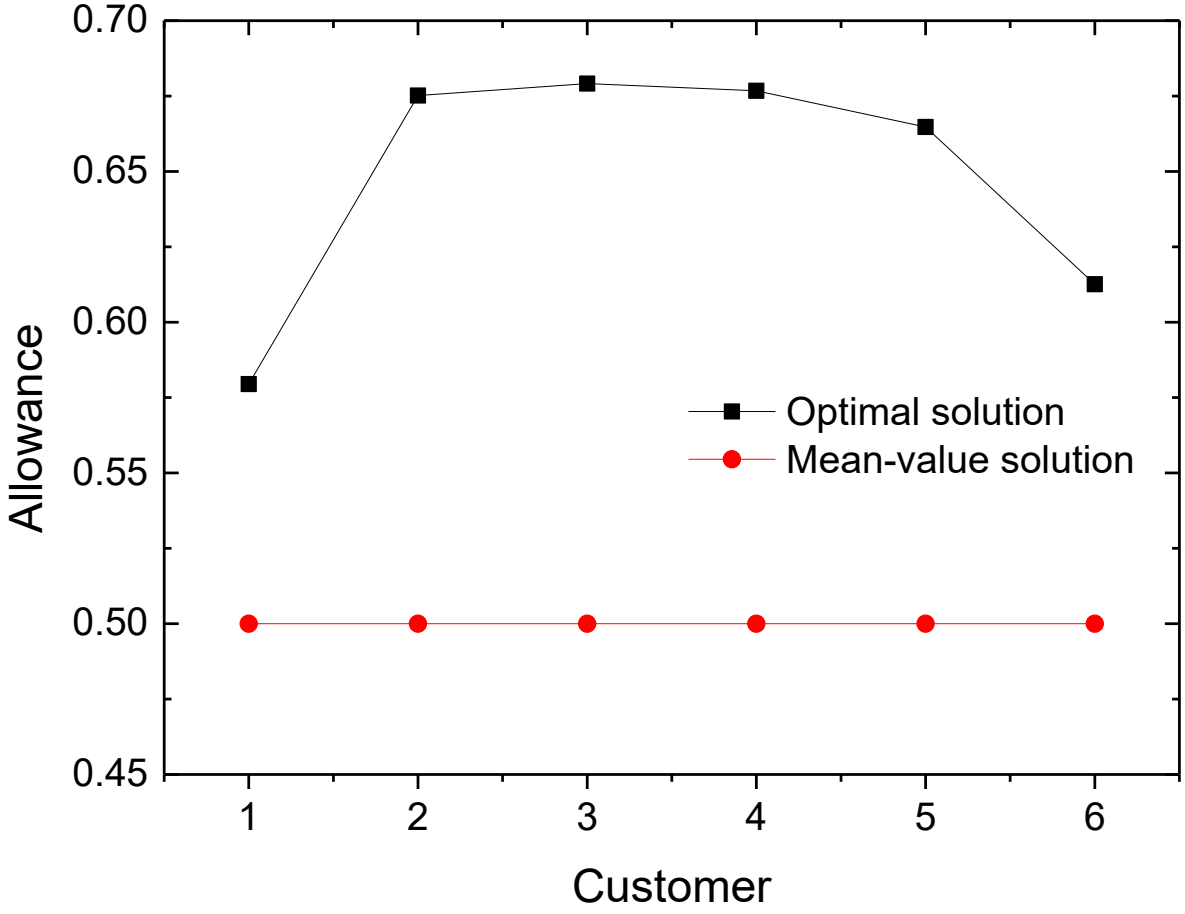
$$Q^*(\boldsymbol{x}^*) \approx 1.651 \pm 0.013$$

$$Q(\hat{\boldsymbol{x}}) \approx 2.204 \pm 0.031$$

VSS=?; EVPI=?

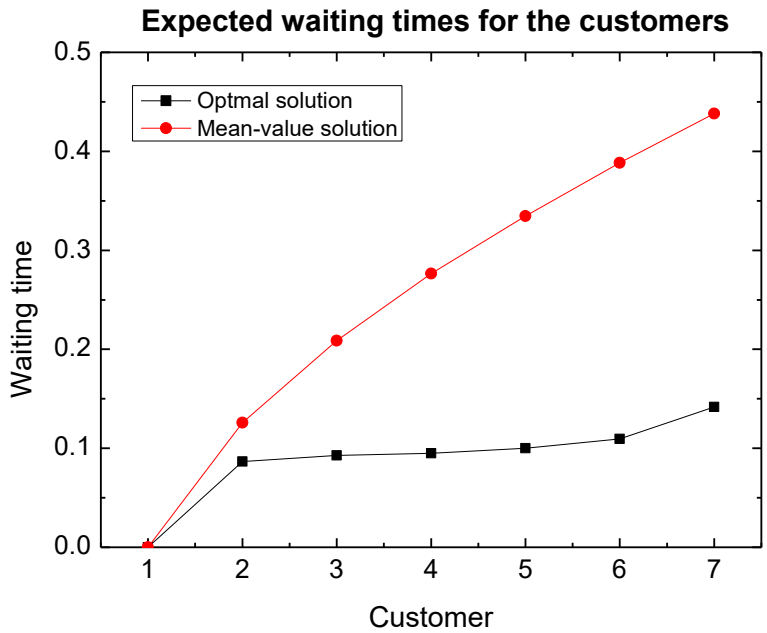
Solutions

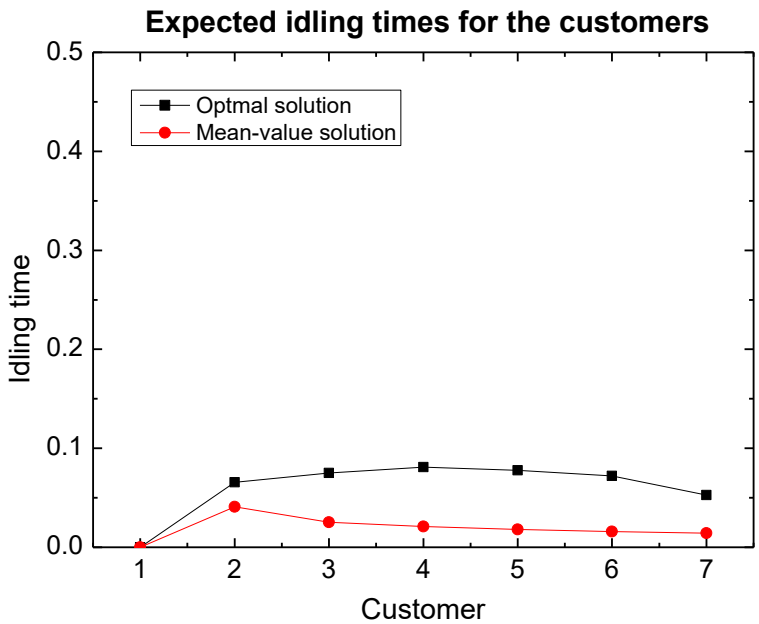
Customers in the middle in the optimal schedule are allotted more allowances.

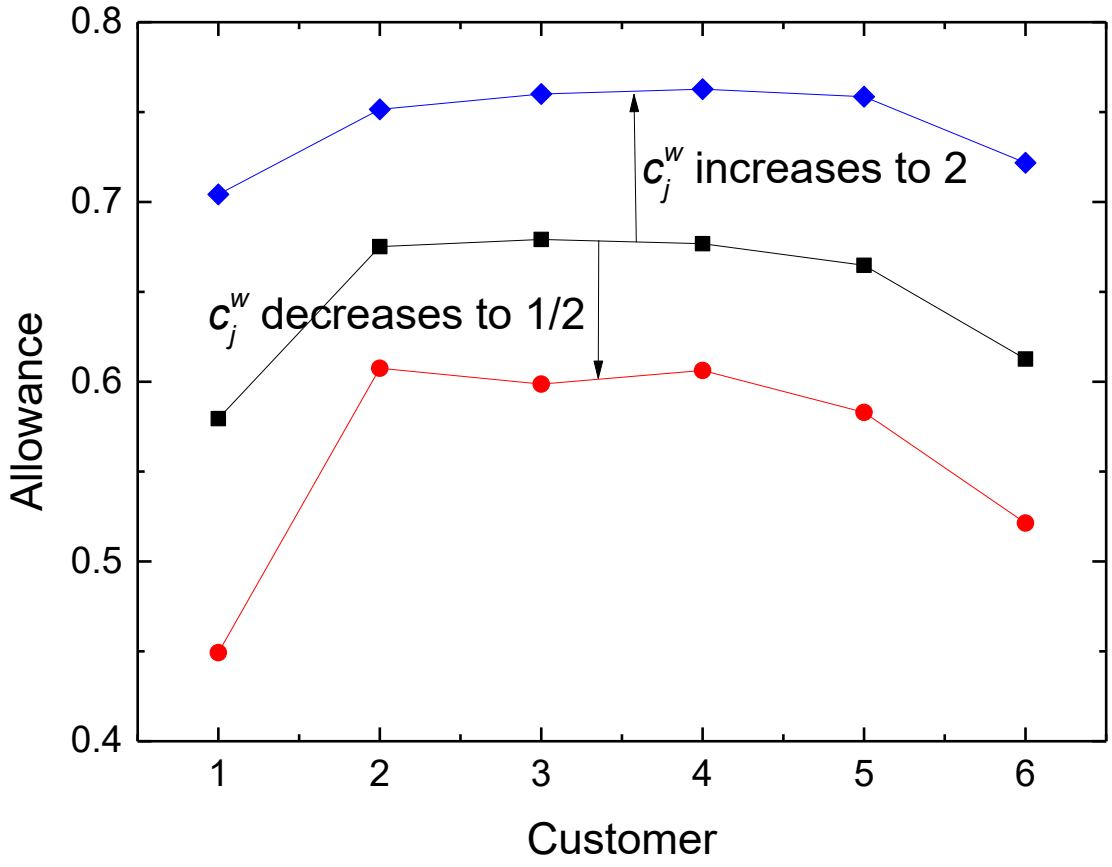


Waiting and idling times

The optimal solution outperforms the mean-value solution in lowering customer waiting time which increases rapidly as the number of customers increases.





Sensitivity analysis

The optimal allowances increase with the waiting time cost.