LPT has the following worst case performance guarantee:

Theorem (5.1.1 in Pinedo): For $Pm \mid\mid C_max$

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3m}$$

Proof:

The proof is by contradiction. Assume there is a counterexample with

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} > \frac{4}{3} - \frac{1}{3m}$$

If more than one such counterexample exits, then there is a smallest counterexample among them. Let n denote the number of jobs in this smallest counter example. Under LPT the smallest job is the last to start and finish. If it is not the last to start and finish then there is a smaller counterexample since we can remove this last job and $C_{max}(LPT)$ will be the same, but $C_{max}(OPT)$ is the same or smaller.

Given the above argument the start time of shortest job under LPT for the minimal counterexample is $C_{max}(LPT) - p_n$. Because all other machines must be busy at this time, according to the definition of LPT, it follows that

$$C_{max}(LPT) - p_n \leqq \frac{\sum_{j=1}^{n-1} p_j}{m}$$

since the right hand side is an upper bound on the start time of the shortest job. Therefore it follows that

$$C_{max}(LPT) \leqq p_n + \frac{\sum_{j=1}^{n-1} p_j}{m} = p_n\left(1 - \frac{1}{m}\right) + \frac{\sum_{j=1}^{n} p_j}{m}$$

Where the rearrangement on the right hand side involves changing the upper limit on the summation from n-1 to n. Putting all this together it follows that

$$\frac{4}{3} - \frac{1}{3m} < \frac{C_{max}(LPT)}{C_{max}(OPT)} \leqq \frac{p_n\left(1 - \frac{1}{m}\right) + \frac{\sum_{j=1}^{n} p_j}{m}}{C_{max}(OPT)}$$

Given that the left hand side must be strictly less than the right hand side we have the following:

$$\frac{4}{3} - \frac{1}{3m} < \frac{p_n\left(1 - \frac{1}{m}\right)}{C_{max}(OPT)} + \frac{\frac{\sum_{j=1}^{n} p_j}{m}}{C_{max}(OPT)} \leqq \frac{p_n\left(1 - \frac{1}{m}\right)}{C_{max}(OPT)} + 1$$

The last inequality is true since the minimum makespan can be no lower than the following (based on the relaxation of $Pm||Cmax$ discussed in class), i.e.,

$$C_{max}(OPT) \geq \frac{\sum_{j=1}^{n} p_j}{m}$$

Therefore we have that

$$\frac{4}{3} - \frac{1}{3m} < \frac{p_n\left(1 - \frac{1}{m}\right)}{C_{max}(OPT)} + 1$$

And rearranging this yields

$$C_{max}(OPT) < 3p_n$$

This implies that makespan is strictly less than 3 times the processing time of the shortest job. Therefore, there are at most 2 jobs per machine for the minimal counter example. However, LPT is provably optimal in this case (<u>this part of the proof is left as an exercise</u>).

Given that LPT produces an optimal schedule for this minimal counter example it follows that the initial assumption:

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} > \frac{4}{3} - \frac{1}{3m}$$

cannot be true and thus a counter example that contradicts the original proposition does not exist.